PETER CARR: So, I welcome comments or questions at any point during this talk. We have an hour and a half, and I have only 50 slides. So we should be OK. So, this is joint work with Jiming Yu, who's a colleague of mine in my group at Morgan Stanley. I head up the global market modeling team at Morgan Stanley. It's a group of about 70 PhDs, mostly, spread around the world. There's about 30 of us in New York, some in London, quite a few Budapest, and a few in Beijing.

The title of this talk is *Can We Recover?* And it's meant as a triple entendre. So, it could refer to either the systemic risk arising from the credit crisis, or the main result in a recent paper by a Professor here at MIT named Steve Ross in the Sloan School, or it could actually be the academic and practitioner reaction to this result. So, it's really about two and three. So, it's not about can we recover from the crisis.

There's a professor at Sloan School named Stephen Ross. And he's very well-known in academic finance. Your professor was kind of to mention that I won Financial Engineer of the Year. And that was two years ago. I was like the 20th winner. He was the second winner. The first winner was another MIT professor, Bob Merton.

So anyway, he wrote a paper a couple years ago, and it's only now about to be published. So this is like typical in academic circles. It takes a long time for a paper to come out. And this paper is coming out in *Journal of Finance.* That's what JF stands for. And *Journal of Finance* is the main journal for the academic finance community. And the title of the paper is *The Recovery Theorem.* And that's also the title of the theorem one in his paper. And that theorem one we'll go over. And it gives a sufficient set of conditions under which, what Professor Ross calls "natural probabilities," at a point in time can be determined from-- OK mathematically, from exact knowledge of Arrow-Debreu security prices, which you probably don't know what they are. But less mathematically, we'll just say from market prices of derivatives. OK, so derivatives you've heard of, I'm sure-- things like options, for example, on stocks or stock indices, could be on currencies.

So, imagine that you look at Bloomberg. Bloomberg publishes a whole bunch of prices. And the idea is that you take this information, and from it you're learning what the market believes are the probabilities concerning the future. And so, if the option is on S&P 500 stock index, then you're learning from options prices what the market believes are the likelihoods of various possible levels for the S&P 500.

So, we take this information on Bloomberg and, truth be told, we use it along with some assumptions to extract these implied market probabilities. So, I want to tell you what those assumptions are. And so, the actual output of this analysis is a probability transition matrix. Or, if you do it in continuous time, you'd call it a continuous state space. You'd call it a transition probability density function. So the key word there is "transition."

And what transition means is you're getting not only the probabilities going from, say, the current S&P level to any one of several levels, but even the probabilities of going from some other level than we're presently at today to that range of levels. You could say, for example, the market believes that given that we're here now with S&P at, say, 1,500, that the probability of more than doubling is one half, for example-- which would be really high. But you know, I'm just picking numbers randomly here.

And you can even say that if S&P were to drop instantaneously to half its level, that the probability of more than doubling from there is, say, one-third. So, you can answer questions like that. That's The output of this type of thinking.

So there'll be three probability measures that we can be thinking about. And we'll call them P, Q, and R. And I'd like to tell you what each of them means. So P stands for physical probability measures. So the P is for physical. And think of that as the actual objective reality of future states for, say, S&P 500. So let's say God knows that, for example, the probability that S&P is up by the end of the year is one half. And we, unfortunately, not being God, don't know that.

But let's say the philosophy is that. There is some sort of true probability of S&P being up at the end of the year. And let's say I used a half. Maybe it's 60%. If it is 60%, then the probability of S&P being down at the end of the year is 40%. And the point is P is meant to indicate the frequencies with which S&P 500 in my example takes on various values.

Now, there's another probability measure that people in derivatives spend a lot of time working with. And that's called risk neutral probability measure, and it's often denoted by a letter Q. So we'll denote it by Q. And the concept of a risk neutral probability measure was also actually proposed by Steve Ross many years ago. And it's called risk neutral because when you're working with it, if you think about how fast prices appreciate over time, then they grow randomly. But on average, under this risk neutral measure Q, the grow at the same rate as your bank balance would grow.

So your bank balance, let's say, nowadays is growing at best at the rate of 1%. And when you look at how fast, historically, stocks have grown, it's actually much higher, on average, than 1%. It's more like about 9%. So we would call the difference between 9% and 1%-- we call that 8% differential risk premium. And let me just pretend there's no dividends to keep life simple when I say this.

So now, this risk neutral measure is kind of a fictitious probability measure in the sense that it's not describing the actual probabilities or frequencies of transitions, it's more a device, or a tool, or a trick that's handy. And one of its properties that causes it to earn the name risk neutral probability measure is that when you look at how fast, say, S&P grows on average under this risk neutral probability measure Q, it would be growing nowadays at 1%-- so the same as your bank balance is growing at. So the word risk neutral is meant to indicate that the growth rate under this measure is consistent with investors in the economy being risk neutral, meaning that they require no premium for bearing risk.

Now there's a third probability measure that we're going to be talking about today that actually you won't find any literature on. And we're going to call it R. It seems like a natural letter to pick, having already gone through P and Q. And you can think of the R as standing for recovered probability measure. And it's going to be the probability measure that we get from market prices as I was talking about earlier. And the operational meaning of this R measure is it's capturing the market's beliefs regarding the future. But we allow for the possibility that the market could be wrong.

So we're applying this to say houses and housing prices in, say, 2005-- it may well be that if we looked at Bloomberg and got prices of mortgage-backed securities, that we would extract an R probability measure that says housing prices are going to continue on their incessant upward trajectory. And, you know, we're going to keep growing at the rate of, say, 15% a year each year for the next 10 years, or something like that. So, that could be what the market's beliefs were back in 2005.

And we know now that those beliefs were wrong, if that was what the market was inferring. So, I want to allow for at least the theoretical possibility that the market could be wrong. And so, that's why I'm drawing a distinction, let's say, between the R probability measure that captures the market's beliefs and the P probability measure that captures physical reality.

So now, there's a lot of people in finance who simply cannot accept the possibility that the market could be wrong. And for those people-- the sort of true believers in market efficiency-- they are free to set R to P every time they see an R. But I want to allow for the possibility that what we recover is not physical probabilities, but simply the market beliefs. And anyway, it's kind of semantics. It's good semantics if the probability measure we recover is the one Ross said we should get. R stands for Ross.

So Ross calls the probability measure that we recover-- he calls them natural probability measures. And well, let's say, that suggests that the risk neutral probability measures are unnatural, which I think is fair actually. Because when you hear the word probability, you tend to think about frequencies with which events occur. And the risk neutral probability measures do not give you the frequencies with which events occur. What the risk neutral probability measures give you is instead prices of so-called Arrow-Debreu securities.

So, let me give you a sense of what that means. So say I tell you that the risk neutral probability of S&P 500 being up at the end of the year is 40%. Then how should you interpret that? Well, you should simply interpret it as this. Imagine that you can agree now to buy a security that pays $1 just if S&P 500 is up at the end of the year. And usually when you and I buy things, we buy them in a spot market. So we pay now for things. But sometimes your credit is good, and you can actually agree now to pay later. So, we're going to be thinking that you're agreeing now to pay later some fixed amount in return for the security that's going to pay $1 just if S&P 500 is up at the end of the year. And if I tell you that the risk neutral probability of S&P 500 being up by the end of the year is $0.40, what that means financially is that you agree now to pay $0.40 at the end of the year for the security.

So, you can imagine there'd be another security that pays $1 just if S&P 500 is down by the end of the year. And the only possible price that that security could have in an arbitrage free world would be $0.60. Because if you were to buy both securities, then you get paid a total of $0.40 and $0.60. So you're agreeing now to pay $1 at the end of the year. And then having both securities, either S&P is up, or S&P is down. And so, you collect $1 from one of them and not the other. So if, for example, the one paying if S&P is up cost $0.40, while the one paying if S&P is down only cost $0.50, then there would be an arbitrage, which we would buy both securities, agree now to pay $0.90. And then get $1 for sure at the end of the period. So we'd be up $0.10 by the end of the year. Question--

AUDIENCE: [INAUDIBLE]

PETER CARR: Yes. It's more than similar. They are digital options. Yeah. So, that's right. So, that's another term, which I'll actually use on the next slide. So, that's exactly right. So, digital options is just too good a term. So economists, in order to obfuscate and look smart, call them [INAUDIBLE] securities.

So, continuing with the obfuscation, I want to tell you about a world with a representative agent. So, economists are fond of trying to formally model the market. You read the newspaper. Every day, you'll read something like market thought that stocks were no longer a good investment. So there was a sell-off. Market is a nice, short word to capture what people are thinking. And so economists, rather than say the market, will say there's a world where the representative agent--

So this representative agent is a fictitious investor who has all the mathematical properties that we give an investor, such as utility, function, and an endowment, and so on. And what makes this particular investor a representative agent is that this agent sort of finds that current prices are such that it's optimal to hold exactly what's available in the amount that is available.

So if what's on offer is, let's say, some Google shares, and some Apple shares, and some IBM shares. And if we take the total market cap of Google, total market cap of Apple, total market cap of IBM, and, let's say, Apple's biggest. I don't actually know whether Google's bigger or IBM, but let's say it's Google, and then IBM. So let's just say Apple's biggest, then Google, then IBM. Well, this investor would actually find that it's optimal for him to have most of his money in Apple, second most of his money in Google, third most amount of his money in IBM, that's the representative agent.

So, he's acting in the way the whole economy is acting. Well, I've been working in Wall Street now since 1996. I have yet to hear a trader tell me about a representative agent. Anyway, so although I understand what the words mean, and even the math, I wanted to present this material in a way that, let's say, at least quantitative traders could understand it. So I tried to get away from representative agents and present these ideas in the language that at least quants on Wall Street are familiar with.

So, I won't be talking about a representative agent, and I will be talking instead about something that's probably not too familiar to you, but at least quants have heard of. And that would be something called numeraire portfolio. And it also goes by other names. Another name is growth optimal portfolio. And it even has a third name, which is called natural numerator. And these are three different phrases that all describe the same mathematical object.

And this mathematical object is a portfolio-- and more precisely, it's the value of a portfolio that has some nice properties. So the growth optimal portfolio indicates one of its properties. This portfolio has a very nice property, which is that in the long run-- meaning over an infinite horizon-- the growth rate of this portfolio is, first of all, random. But second, if you take the mean of that random growth rate, that mean is actually the largest possible among all portfolios.

So, starting with Kelly in 1956, this particular portfolio with the largest mean growth rate over an infinite horizon receives a lot of attention. It's actually quite humorous, some of this attention that it's received. So, Kelly was a physicist who worked at Bell Labs. And he was actually a colleague of Shannon's at Bell Labs. So Shannon did his seminal work at Bell Labs, but actually came here after that.

And his ideas really caught on-- and especially, I'd say, started the field of information science, we'll call it-- whatever. But Kelly was applying these ideas to finance. And certain financial economists we're lesson enthused about how the application information theory to finance. So, in particular, there was a financial economist here named Paul Samuelson who championed, I guess, the opposition to this Kelly criterion it's called. And so, I'll just tell you a short story.

AUDIENCE: Excuse me.

PETER CARR: Yeah.

AUDIENCE: If I could just interject--

PETER CARR: Yeah, sure.

AUDIENCE: We had mentioned in an earlier class the book *Fortune's Formula*. And this book goes into a lot of background and storytelling about this whole era and exchanges.

PETER CARR: That's true. It's a fantastic book. I read it. I loved it Especially if you're at MIT, you should definitely read this book. It talks about a lot of MIT professors, some of whom are still here, like Bob Merton. It's a quick, easy read. You don't even have to have a background in finance to really enjoy it. So you can read about the story I'm going to tell you now in that book.

So the story is Samuelson grew a little tired, I guess, with trying to explain to these dumb information theorists that this Kelly criterion was not so great. So he published an article in a journal called *Journal of Banking and Finance*-- that's actually a finance journal-- where he explained why it wasn't necessarily such a good idea to hold this portfolio. And in this article, every word he used was of one syllable, except the very last word of the article, where he managed to say that he has-- I can't even do it in one syllable-- OK, so just ignore my multi-syllabic words.

But anyway, he says, I have managed to write an article with all words with just one syllable, except for this last syllable-- OK, I lost it-- sorry. But anyway, the last word in his thing was syllable itself, which is multi-syllabic-- or whatever. So anyway, it was kind of insane. So, let's move on. So this talk-- it has six parts. And we have an hour to go. So let's say we'll try to spend 10 minutes on each.

AUDIENCE: [INAUDIBLE]

PETER CARR: Yes. Well, that's a good question. So, it does have risk, first of all. It does have a lot of risk. It's not the riskiest, though. So some risk does not carry with it expected return. And so that's why it's not the riskiest-- but it's risky. So Samuelson's objections were precisely what you're getting at, that this is a fairly risky strategy. So, I'm glad you brought that up.

OK. So there's six parts to the talk. I'm going to go over what Arrow-Debreu security prices are-- so again, they're digital options prices-- and their connection to market beliefs. I'll talk about this Ross recovery theorem. So in Ross' paper, which you can get on SSRN, he does everything in a setting that's called finite state markup chains. And so that's mathematically simpler than what we use in practice. And I totally agree that when you try and introduce something, you do it in the simplest mathematical setting.

So now that he's done that, I wanted to do it in a more familiar setting, which is a diffusion setting. A diffusion has an unaccountably infinite number of states. And I still want to keep things as simple as possible while going beyond finite state markup chains. So I work in a univariate diffusion setting. So there's only one source of uncertainty, which is the same as in Ross. And our technique is to get these results. It's based on something called change of numeraire. So numeraire is a technical term, actually, that describes an asset whose value is always positive. So there are securities whose values can have either sign.

So, swaps are a classical example. So a swap is a security which at inception has zero value, actually. And then the moment after inception, the world changes, and the swap value either becomes positive or becomes negative. So a swap would not be eligible to be a numeraire because of that property that its value is real.

On the other hand, if you take a stock, its price is always positive-- well, that's debatable actually-- so let's say let's not do stock. Let's do a treasury bond. A treasury bond-- US Treasury bond-- its price is always positive. The reason I want to shy away from stocks is because we take Lehman Brothers stock, for example. It's price was positive, then became zero. And actually, because Lehman's price became zero, Lehman's share you could not be a numeraire. So when I say that the numeraire value has to be positive, I mean strictly positive.

And so anyway, there's this literature about how to change numeraire, how to go from one asset with positive value to another asset with positive value. And it's useful for understanding how this Ross recovery works. So, we apply it when we have a so-called time homogeneous diffusion. And I'll tell you what that means-- over a bounded state space. So bounded state space means that the set of values that the diffusion can take is in some finite interval. So if you're thinking about the uncertainty being, for example, S&P 500, then the natural lower bound for S&P 500 would be zero. And you have to accept that there's a finite upper bound in order to apply our results.

Now you know, personally, I have no problem saying the S&P 500 is bounded above by 20 trillion. OK, but some economists have actually said this is ridiculous. and challenged my work, and stuff like that for that assumption. So, because of those challenges. I have actually been trying to extend our work to an unbounded state space, where, let's say, the largest possible value for S&P 500 would be infinity. And I've found, actually, that it's not that easy. And so sometimes, I can make it work, and sometimes they cannot. So, when we get there, I'll explain some examples that work and some examples that don't.

So this last section is kind of incomplete, this sixth section. And so, basically, I've got examples that fail, examples that succeed. But I don't have a general theory. So there'll be different assumptions in different parts of the talk. But within a section, there's only one set of assumptions operating,

AUDIENCE: Excuse me.

PETER CARR: Yeah.

AUDIENCE: [INAUDIBLE] the value of anything is [INAUDIBLE].

[INTERPOSING VOICES]

PETER CARR: That's been my response too. So the universe is bounded. And it's growing, but it's bounded. So, I agree. You know, I'm on your side on this. I'm just telling you what I've been told. Yeah. So, I'm working on it anyway, just so they can shut up. But, anyway--

AUDIENCE: Actually, I have some comments on the issue of the numeraire. You'll tell me how connected this is-- but with the Kelly criteria, one of the origins of that is if you have a gambling opportunity where it's favorable, how much of your bankroll should you bet on that gamble? And basically, the Kelly criterion tells you what proportion of your bankroll you should invest at all times. You should never bet everything. And if you do bet everything, you lose everything, and you're done. So, the issue with the numeraire portfolio and never being able to go down to zero, in the sense that you can never go bankrupt. And so, assumptions of being able to always rebalance your portfolio--

PETER CARR: So, just give you a flavor of what this numeraire portfolio is-- you're betting a constant fraction of your wealth in every security. So let's just keep it simple. There's only two securities. One is risky, and the other's riskless. And so you might be betting putting 40% of your wealth in the risky one, and 60% then in the riskless one. And that's when you start. So you have $100, and you put $40 in the risky one, and $60 in the riskless one.

And then, time moves forward. And let's say the price of the risky one changes. Then when you revalue using the new price, it's unlikely that 40% of your wealth is in the risky one. So in fact, if the price went up of the risky one, you'll have more than 40% of your wealth in that risky one. So you need to sell some of that risky one. And then the money you get, you put into the riskless one. And so, every time the price changes, you need to trade, theoretically, in order to maintain a constant fraction of 40% of your wealth invested in this risky asset.

So we assume zero transactions cost when we do this analysis. Because there are positive transactions cost. One should take that into account. And there is literature on how to do that. So, I won't be formally entertaining transactions cost in this talk. There's work here at MIT, actually, on doing that. For the question of how should you invest, it feels like it's a complication that won't change anything qualitative about-- it'd definitely change how frequently you trade, but it wouldn't, let's say, it's unclear how it would change your initial investment across bets.

So, let's begin with part one. So we have the digital options, or also called binary options. That's another term. And they trade, actually, in FX markets-- so foreign exchange. And they pay one unit of some currencies, so say dollar-- If an event comes true. So it might be that you're looking at dollar euro. And if by the end of the year, dollar euro exceeds $2, then you get $1. Otherwise, you get $0.

So there would be a price in the FX markets. And it would be a spot price typically-- so meaning you have to pay now for it. Let's let A, for arrow, be the price today of such a security. And the subscripts on A are J, given I. So, the idea is that you can think of yourself as in a finite state setting. There's various discrete levels of say, dollar euro that we have that can be possible today. And there's also various discrete levels for dollar euro by the end of the year. And I indicates the state we're in. So maybe dollar euro is $2 per euro right now. And J indicates the state we can go to. Maybe we can go to $3 per euro.

So in my example, [? A32 ?] would be the price of an Arrow-Debreu security, given that the current dollar euro exchange rate is $2 per euro, and it pays $1 just if dollar euro transitions from $2 per euro to $3 per euro. So the idea is we have discrete states. And let's say these are values that are possible at the end of the year. And the example I just went through-- you're getting $1 just if-- is $3 per euro at the end of the year. So the height of that vertical line is one.

Now, I'll just comment that this is a slightly exotic option, in the sense that-- let's call it exotic. It's slightly exotic. So in contrast with exotics, there's this term "vanilla." OK, and it actually indicates a flavor of ice cream. So, we have this terminology which you get used to after awhile. And you can't understand when you talk to a man on the street why they don't understand what a vanilla option is. So a vanilla option is a payoff that looks like this-- so it's a hockey stick payoff. And that's the payoff from a call option. And it turns that there is a portfolio involving options at three different strikes that can perfectly replicate the payoff to this Arrow-Debreu security. And so, here is a payoff from a single option struck at two. And I'll just say that if I had changed the strike to, say, be three, then it would look like that.

Now, you can combine options in your portfolio. So you could, for example, buy a call struck at two. And then you can furthermore sell two calls struck at three. So if you sell, on top of that, two calls struck at three, you end up creating a portfolio that goes like this. And so, they can go negative in value. So if you not only buy one call struck at two, sell two calls struck at three, but furthermore, buy one call struck at four, then you end up with this payoff, which the payoff is called a butterfly spread payoff. Because the picture is meant to remind you of a butterfly. And notice that if the only positive values before the FX rate were $1 per euro or $2 per euro, or $3 or $4, or $5, if that were the world. Then notice that when you formed that portfolio, the only positive payoff you can get from it is $1 just if dollar euros at 3.

You can synthesize a Arrow-Debreu security using a butterfly spread. So, this was pointed out many years ago. So even if the FX market were, let's say, not directly giving us the prices of digital options, we could from vanilla options extract the implicit price of a digital. And what you would learn from vanilla options is what the market is charging for the digital, given that, let's say, we're presently at $2 per euro. And what you would not learn from these options prices is what the price of the security will be should we today have the exchange rate change to some other value. However, you can make assumptions as have what the options prices will be were today's exchange rate different. So, that's commonly done in practice.

So a common assumption, for example, is that the probability of transitioning from two to three-- so moving up by half-- so you're moving up by half of two to three-- is the same if you were at any other level. So for example, if you were at four, then the probability of going to six would be whatever the probability is of going from two to three. Because if you're at four, the probability of going up by half of four to six-- that's the assumption. OK, so that's called sticky delta, and it's a common assumption.

So if you make that assumption, then you can take the information at just today's level. And like, let's say, you know, all the digital's from two, and you can make that assumption. Let's say the probability of a given percentage change is invariant to the starting level. And then you can, from that, figure out what the probability of going from four-- a different level than we're at today-- is to all these different levels. So you can go from a vector bit of information that the market is giving you to a matrix. And that matrix is called transition matrix. And so, we're going to, in this talk, assume that somebody's made such an assumption. And so, you actually know this matrix. So you actually know as a starting point what the prices are of these Arrow-Debreu securities or binary options starting from any level and going to any level.

I think in order to get through my whole talk, I'm going to skip these slides. Because they're kind of like just being very precise about what some terms mean that aren't going to be that important for the overall story. So OK, let's go to this slide. So we think of there being just a single source of uncertainty x, which could be dollar euro. And we imagine that we have this matrix of Arrow-Debreu security prices. We know every number in this matrix. And we ask what does the market believe about transitions from any place to any place? What does the market believe is the frequency of these transitions?

Now, suppose that the number that's indicating the price of the Arrow-Debreu security going from two to three-- suppose that number is, say, 0.1. Now what does it mean? It just means that you pay $0.10 today for security paying $1 just if you go from two to three. That's all it means. Now you can ask what is the frequency with which you go from two to three? It need not be 10%. There's at least two reasons why the $0.10 price could differ from the probability of going from where you are to where you get paid. One such reason is simply time value of money.

So if you were to buy all these Arrow-Debreu securities, [INAUDIBLE] paying off one for every state, you'll find that the total cost is less than 1, even though the payoff, for sure, is one from the portfolio. And that's simply because of the time value of money. So when you put $1 in the bank today, you actually get more than $1 back when you pull out at the end of the year. And if you do the inverse problem-- how much do you have to put in the bank today in order to have $1 at the end of the year? It might be $0.95. So that's called time value of money. And so, just the fact that you have to pay now for the Arrow-Debreu security. And you only get paid off at the end of the year. That causes this price of $0.10 to be lower. So that's just discounting for time. The interest rates are positive.

So that's one effect. Now there's another effect, which is called risk aversion. So risk aversion is the thought that even if the interest rate was 0, to abstract away from the effect I just described, that it still may be the case that a $0.10 price paid for an Arrow-Debreu security transitioning from two to three is different from the probability of such a transition, the real-world probability of such a transition. Because, for example, it may be quite desirable to get money in that state, in which case $0.10 is over the real world probability. Or it could be the opposite that maybe it's not desirable to get money in that state, in which case $0.10 is under the real world probability.

So give you a more concrete example-- let's say something that is maybe a little closer to home is let's say this is S&P 500-- and I know the values are very different than the numbers I'm indicating here-- but let's just forget about the actual numbers. So the point is let's suppose that it's equally likely, in terms of true probabilities, to go from two to three as it is to go from two to one. So we have two Arrow-Debreu securities. One struck at one. The other struck at three. And I'm telling you that it's equally likely that you go up by one as it is to go down by one.

Now you can ask the question does it necessarily mean that the prices of these securities that pay $1 are the same? And the answer is no, not necessarily. And actually, the sort of standard thinking in financial academic circles is that for S&P 500, it would cost more to buy this Arrow-Debreu security than it would cost to buy that one, even though everyone agrees that it's equally likely to get paid from each of them. And the reason that it's thought to cost more to buy this one than it is to buy that one is because this one has an insurance value. So the thinking is that on average, people are long the stocks in the stock market, and that that means that they're really upset when the stocks fall. And so they really like this one that ends up paying should the stock market fall from two to one, whereas this one, while it's nice to get money, let's say you're already fairly wealthy from the fact that you're owning stocks and the stock market went up. So you'll pay a positive amount for this security, but not as much as you pay for this one.

So that's called risk aversion. So what we want to do is go from the prices that are contaminated, let's say, by time value of money effects and by risk aversion effects. And we want to cleanse them of that contamination and try to extract what the market believes are the frequencies of the future states. So I'll tell you that this was thought to be impossible before the Ross paper, and in fact, without making assumptions, it is impossible. So all Ross did is make some assumptions that are thought to be fairly mild by some, including me. And so he essentially, in essence, showed the power of some assumptions. That's one way of thinking about it.

So again, let's denote by R the recovered probability measure which will tell us the market beliefs about the frequencies future state. And we don't know R when we start. What we do know is these Arrow-Debreu security prices, I'm assuming. And we'll denote those by A for Arrow. So what Ross' paper does is it says, you know A. And if you're willing to make the following assumptions, then you'll know R. So what are the assumptions? Well, before I tell you assumptions, I have to tell you some terminology so that you understand the assumptions.

So he'll work with a pricing matrix A, which we've actually been going through. So that's the Arrow-Debreu security prices index by starting state and final state, which we'll call x is starting state, y is final state. Then there'll be the desired output from this analysis, which he calls natural probability transition matrix. So these are the markets beliefs for every starting value x and for every final value y. And then there'll be something called pricing kernel, which is literally the ratio of these Arrow-Debreu security prices to these output natural probabilities.

So, if you want to get an understanding of what this pricing kernel is, you can think of it as an attempt to capture the effects from time value of money and from risk aversion. So think of it as a normalization. You start with A, and A is actually affected by three things. It's affected by the unknown real world probabilities-- or at least markets beliefs of them. A is also affected by a second thing, which is time value of money. And A is affected by a third thing, which is risk aversion. So if we take A and divide by P, then we're normalizing for the first effect, the frequencies. And so we're left with just the combined effect from time value of money and from risk aversion.

And so, let's say, if interest rate were zero and people were risk neutral, then we would actually expect A to equal P. And so this ratio would be just constant. So Ross talks about a world where the representative investor, and essentially, this is an assumption-- this equation you're seeing here. It's an assumption on the form that a function of two variables takes. So phi, first of all, is a positive function. So phi is positive, as opposed to so phi cannot take negative values. So both A and P are positive. And phi is a function of two variables-- x and y. And what this assumption is doing is it's saying, well, let's put structure on this function phi because it'll help us to find it if we put the structure.

So this is the first key assumption actually-- that the function of two variables x and y actually has the form on the right, which, for a moment, just ignore the delta for a moment. And then you can see that what you have on the right if you ignore delta, if you think of delta as one, is you have a function of y. And then you have the same function of x. So it's written in a convoluted way with this U prime, and C, and all that stuff. But if delta's one, then you have a fraction whose numerator is a function of y and whose denominator is the same function, but of x.

In essence, what that does is it reduces the dimensionality of the thing we're searching for by a lot. So we started by searching for a function phi of two variables. And we, by this assumption, reduced the search to a function of one variable, which is, say, the function in the numerator, which is the same as the function in the denominator. So, now let's bring back delta. And delta's a scalar here, and it's a positive scalar. And so we need to search for that as well. So in the end, we reduce the search to a function of one variable and a scalar delta.

So the economic meaning of, first of all, the function of one variable as it's called marginal utility. And it's meant to indicate how much happiness you get from each additional unit of consumption. So it's the typical-- what we think it looks like as a function of c-- U prime as a function of c is thought to typically look like that. So it's positive, meaning every unit of consumption makes you happy. And it's actually declining, meaning the first unit of consumption makes you real happy. Then the next unit of consumption still brings some happiness, but not as much, and so on. So that's the kind of function we're looking for. U prime is a function of c. He won't actually find U prime is a function of c. He'll find the composition of U prime with a function c of y. Keep that in mind.

Then, there's that delta. And that's, again, a positive scalar. And it's meant to capture time value of money. And so, that's like the y is the state at the end of the period. And x is the state at the beginning of the period. And so, that's why delta's associated with the numerator, not the denominator. So delta would be a number like 0.9. And that indicates how much discount you give to, let's say happiness received in the future, rather than now.

Now, here's a quote from Ross' paper that is his Theorem 1. That's called the recovery theorem. And the only thing is I changed the letters to conform with the letters I'm using, rather than the ones he used. And that's because his choice of letters is completely unnatural to me and most people. So I don't even want to tell you what he used. So anyway, whereas I tried to choose letters that make sense. So I used A for Debreu--

So anyway, he says, you have a world with a representative agent. So that's actually this restriction that we talked about on the last slide. And then he says, if the pricing matrix-- which is the Arrow-Debreu security prices-- is positive-- which means that all entries in it are strictly above zero-- or irreducible-- which means that some entries have zeros, with the rest being positive, and there's some structure, which we need not get into where the zeros are-- then there exists a unique solution of the problem of finding P, which P is actually market beliefs. And I've been calling that R often.

So anyway, I slipped a bit there and called it P. So anyway, that's market beliefs about the frequencies of future states. He'll also get as an output the delta, which is the positive scalar telling you the market's time value of money. And finally, this pricing kernel phi, which is the ratio of A to P. So, what you're supposed to realize, even though he didn't say it, is that as a result-- well, OK. So he did say it actually. You're finding P. I think that's the main thing. He's actually saying, if you make these assumptions, surprisingly, there's only one possible real world or market beliefs that are consistent with the data and the assumptions made.

To give you a sense of what the importance of this result is-- so prior to his paper-- I mean, people have been interested in trying to infer from market prices what the market believes. But they always thought that you had to supply some parameters that capture market risk aversion. So for example, common approach is to assume that you have a representative investor. And that they have a particular type of utility function called constant relative risk aversion. And there's a parameter in that utility function. And you had to specify the numerical value that parameter takes before you could learn the market's beliefs from prices. And no one ever felt very comfortable specifying that parameter.

So what Ross essentially did is he managed to essentially do the identification non-parametrically, where you don't have to suppl any parameters. And so you essentially just have to buy his assumptions. You don't have to do any work to actually go from market prices to market's beliefs. OK, so let's skip these remarks. Yeah.

AUDIENCE: Can you elaborate on the fact that risk aversion does enter in?

PETER CARR: Yeah. So the exact statement is you don't have to supply a parameter that describes the amount of the market's risk aversion. Rather you have to accept this assumption-- and I'll show you-- this assumption about the structure of phi. OK, so if you just accept that this function of two variables doesn't have the full amount of degrees of freedom that an arbitrary function two variables has, it has a reduced number of degrees of freedom implicit on the righthand side. So remembering that x is actually just a vector of finite length and so is y, then think of the lefthand side as having degrees of freedom n squared. And on the righthand side, you're looking for the numerator function is just a vector of length n. And the denominator function is the same function, so the same vector. And then there's also this delta.

So let's say on the lefthand side, you're describing something that without restriction is of order n squared. So let's say n is 10, so it has 100 degrees of freedom. And on the righthand side, you're describing a vector of length 10 along with a scalar-- so 11 degrees of freedom. So you have to accept that you're willing to before you place any restriction, it's 100 degrees of freedom. Now you make your restriction. It's 11. You have to accept that. And if you do, then he'll tell you the 11 entries.

That's it. So you don't have to supply anything. So I haven't told you how we'll find them. That's probably what you're asking-- how the hell will you get to 11? OK, so I haven't shown you that. Yes?

AUDIENCE: Just really quickly, the c change as a function of time and [INAUDIBLE]?

PETER CARR: c is not a function of time, to answer your question. And then, the argument of c could be a price. It's allowed to be a price. OK? So, that's how you should think of it. So there's a lot of time homogeneity in everything he does here. So he'll never let anything depend on time, actually, to answer your question.

So, I still haven't shown you how he did it. He uses [INAUDIBLE] theorem. I don't actually have slides on how you actually calculate the 11 entries. So I think I just have to refer you to the paper. But he relies on something called [INAUDIBLE] theorem. And I'm going to show you how we-- my co-author and I-- actually calculate the analog of that 11 dimensional unknown. So we're going to work in a continuous setting, where instead of looking for a vector and a scalar, we're going to look for a function, and a scalar, and a function of one variable. So you'll get a sense of how to do it from ours. And essentially, if you discretize what we do, you'll get what he did.

Let's forget these remarks, and let's forget these.

And so now, we'll get into some theory about changing numeraire. So this is a backdrop to how my co-author and I proceed. So again, a numeraire is a portfolio whose value is always strictly positive. And there is a well-developed theory in derivatives pricing about how to change the numeraire.

We're going to use that theory to understand what Ross did. So we start with an economy with a so-called money market account. And so that's a theoretical construct that's pretty familiar to most of us, and it's a bank account. So we're going to be working now in continuous time. So imagine that time, which is continuous, is on this axis. And then we're sitting here today, and we put some money into the bank.

And being poor, we only put $1 in. So then we ask, looking forward, how will this money in our bank change? Well, they do still pay a positive interest rate, and it's awfully small, but it's positive. And so it'll go up. And they change the rate actually. So now, maybe it's 0.5%, but next week, Chase might decide to give you 1%, which it goes up faster. And then they might the week after give you 2%, it goes up faster. Then they might go back to 0.5%. So that's one possible path for your money market account balance. And we don't know the future.

We know how much we're getting over this first little bit of time. But they could actually decide to pay less over the second period, and then the third, or something like that. OK, so it's increasing and it's random. So that's the money market account balance. It's considered as an increasing, random process. And actually, there's nothing in the math that requires it to be increasing if some really cheap bank-- like Bank of America tried this actually-- charge a negative rate. Then it would actually go down with a negative rate. But it wouldn't go negative. So it's still counts as a numeraire. So anyway, that's allowed, as an aside.

OK, so we've got this money market account. So the growth rate is called R, and that's just real value. And then we also have risky asset. So we'll have a total of n risky assets. And then we're going to say there's no arbitrage between the n risky assets and the one money market account. The idea is that we look at Bloomberg's prices for these n plus 1 assets, we're able to extract the Arrow-Debreu security prices. That's the idea. What I'm assuming is that what we're extracting is consistent with the idea that the uncertainty that's driving everything here is a diffusion, meaning that the uncertainty has sample paths that are continuous, but they're allowed to be fairly jagged. So diffusions actually have continuous but non-differentiable sample paths.

And we're going to assume that. So this is a common assumption. This basically got its start here at MIT. And diffusions were first used in a finance context back in 1965 when both Samuelson and McKean were here. So McKean is a probabilist. He's now at NYU where I teach, and he's still active. And diffusions are widely used. So they really got a big boost in 1973 when Black-Scholes and Merton, who were all here, used the diffusion to describe the price of a stock underlying an option. And since then, they've just been used extensively in finance. So Merton, who's here, really, I'd say, pioneered the use of them in finance.

So there's this uncertainty x is probably mysterious to you, hence the name x. So it's like, you get to choose what it is, is kind of the idea. So this is theory. And it's not trying to be overly specific so that you can apply it in different contexts. But you'd like to know at least some examples, I'm sure. So one example would be x is the level of S&P 500. A different example would be x is actually an interest rate. So let's say the benchmark 30 year yield-- x could instead be a shorter term interest rate, something called IOS-- overnight index swap-- is a possible choice for x. When I apply Ross' stuff, that's how I choose x as a short-term interest rate.

In general, let's say I developed a theory that says the short-rate of some function of x. And when I actually apply it, the function is the identity map. The mathematics says that if there's no arbitrage, then there exists-- as we're assuming-- then there exists this so-called risk neutral probability measure that I talked about earlier and denoted by Q. It's related but not equal to the Arrow-Debreu security prices.

So if you were to just imagine that instead of buying these Arrow-Debreu security prices in a spot market, if you instead bought them in a forward market where you actually pay when they mature, then those Arrow-Debreu security prices in the forward market would be Q. So Q and A are really close. So the measure A need not integrate to one. Unless you just do the time value of money, and that's because you're paying in the spot market. If you're actually paying in the forward market, then you don't have to worry about time value of money. And so then, the measure Q does integrate to one. So that's why we call it a probability measure.

Under this probability measure Q, the expected return on all assets is the risk free rate. So that's what that actually says, although you're probably not seeing that this is literally the expected return-- well more precisely, it's expected price change. So the expected price change is what that means is expected price change is the risk-free rate times the price. That's what that says. So if you divide both sides by the spot price when it's positive, then you'll get the expected return is equal to risk-free rate. And we're doing things in continuous time here. So we're working with diffusions.

And you may or may not have been introduced to diffusions at this stage in your mathematical career. But mathematically, one way to describe diffusion is via the infinitesimal generator. So this is a differential operator that's first order in time, second order in space. And let's just say this is formally how mathematicians think about this type of thing. What I've drawn here is a single sample path of diffusion. There's definitely possibility of other sample paths. These actually are an infinite number of paths. But they're all continuous and nowhere differential.

I want to just kind of give you a flavor of how you change numeraires. So we started with the numeraire being the money market account-- this guy. And the idea is we're going to switch to a different numeraire. What we're mainly interested in figuring out is what are the drifts of assets when we measure their values in a different numeraire. So I've kind of given you a sense of what this is about. So you could hold IBM, and every time you get a gain, you could put that gain in your local bank-- Chase-- and see how fast your bank balance grows as you're putting all your gains in IBM in the bank. And you'll get a certain growth rate from that strategy.

Now, you could try a different strategy where you take your gains from IBM. And you actually ship them off over to a British bank, which is denominated in pounds, and see how fast that bank balance grows. And there's no reason that the two bank balances-- the American one and the British one-- need to grow at the same rate. Because they're denominated in different currencies. So we're basically interested to know, given that we know how fast, let's say, the American bank balance would grow, we want to know how fast the British bank balance would grow. And what affects the growth rate of the British bank balance is the covariance, actually, between the dollar pound exchange rate and IBM.

So remember, we're investing in IBM and we're putting gains in either an American bank or a British bank. So IBM stock prices in dollars. And so there's no issues with putting IBM's gains in an American bank. But there's actually a subtle effect that happens when you put IBM's gains in a British bank, which the subtle effect is there's this random exchange rate dollars per pound. And suppose that there's some correlation, for whatever reason, between dollars per pound and IBM.

So suppose the correlation's the following form-- every time IBM goes up, the dollar gets weaker against the pound. So in other words, what happens is IBM goes up, you go hooray, I'm rich. I got all these dollars. I'm going to go put them in a British bank account. But suppose, unluckily for you, every time IBM goes up, the dollar weakens against the pound. And so, you cannot buy so many pounds as a result.

So contrast that with the opposite situation where when IBM goes up, the dollar strengthens as opposed to weakens. Then you can buy lots and lots of pounds with your IBM gains. So the correlation between the dollar pound exchange rate and IBM affects how fast your British bank balance would grow. And that's actually like the key point. So this would be well-known-- especially an FX client.

So what we're actually going to do is find a numeraire such that the growth rate of the balance in that numeraire is actually the real world drift of the underlying. So the idea is let's say that I told you at the beginning of this talk that historically stocks grow at 9% on average. Our starting point here in this part of the talk is that we're starting from this risk neutral measure Q, which, by definition, is the property that stocks would grow only at 1%.

So what we're actually going to do is go find some numeraire which will be correlated with the stocks, such that when we put our stock gains in that numeraire, we end up growing at 9%, rather than 1%. That's the way we think about things. And the key is to find that numeraire that has that property.

I'm going to go fast now-- there's a paper by John Long where he shows that that numeraire that converts a risk-free growth rate into the real world growth rate always exists. And he gave it a name, and he called it numeraire portfolio. It has another name-- growth optimal portfolio-- that Kelly was talking about. So there's a reference if you're interested in following up on this material. So the theory says that there always exists this numeraire called John Long's numeraire portfolio, such that if you park your gains in this numeraire, you end up growing at the real world drift. And so, let's say all we got to do to find that real world drift is go find this special numeraire.

So this part of the talk is about making some assumptions that lead to an identification of that particular numeraire-- John Long's numeraire. We're going to continue to work with diffusions. And now we're going to also impose time homogeneity like Ross was doing. So let's say when I was just talking about numeraire, I was allowing time inhomogeneity.

But now we're going to go time homogeneous. I haven't really been introducing the notation, but AXT is the diffusion coefficient of the state variable x. And now it's just being assumed to be a function of x only. So BQXT was the drift coefficient of x. And now it's a function of x only. RXT was the function linking the short interest rate to the state variable x. And now, it's a function of x only. And finally, sigma LXT was the volatility of John Long's numeraire portfolio. And again, that's a function of x only.

So anyway, another assumption that we're going to impose now in order to determine uniquely what this numeraire portfolio value is is to require that the diffusion that's driving everything live in a bounded interval. So essentially, the sample paths all have to be bounded below by some constant, which could be negative, and have to be bounded above by some constant, which again could be negative. We make all those assumptions, and we move on.

And so in the end, what have we been assuming? So we're assuming that there's a single source of uncertainty x. And it's a time homogeneous diffusion. So that's this middle equation here. And so that says changes in x have a predictable part, which is BQX DT. And they have an unpredictable part, which is A of XDW. So W there is standard Brownian motion. And since I'm big on mnemonics, you might ask why does W stand for standard Brownian motion? And that's because W actually stands for Wiener process-- Norbert Wiener being an MIT mathematician. And the W is a standard notation for this kind of thing.

As an aside, when Bob Merton was here working out all this stuff for the first time in the late '60s, he knew the standard notation for standard Brownian motion was W. But it turns out in finance, the standard notation for wealth is also W. And he wanted to work on stochastic wealth dynamics. And so he had to choose should I use the letter W for wealth, or should I use the letter W for Wiener process? And he chose W for wealth, which meant he had to pick a different letter for Wiener process. And so he actually chose the letter Z. And you'll have to ask him why he chose that letter, because it doesn't stand for anything as far as I know, except that actually the sample paths of a Wiener process look very jagged, so if you turn your head, you might be able to see a Z.

So another assumption is that we're going to restrict the possible dynamics of the numerator portfolio's value. So we're going to let L denote the value of this numeraire portfolio. And the mnemonic here is that John Long invented this concept, so we're calling it L for long. Now it's unfortunate that the inventor of this concept was named Long, actually. Because in finance, the word Long indicates that for a security with a non-negative payoff, if you're long, then you're going to be receiving that payoff. As you pay money now, you're going to receive that payoff. It's the opposite of short, where if you're short a security with a non-negative payoff, then actually you get money now and you have to deliver that payoff later.

So as it happens, this numeraire portfolio has multiple positions in it. And the signs of the positions are allowed to be real-- so positives and negatives. So it's kind of a misnomer. I say Long's numeraire portfolio, and everyone thinks the position's in them are all positive. It's not true-- so the real value. The kind of problem here is that we've put the structure on the value L, John Long's numeraire portfolio, namely that L is a continuous process, but it's not quite a diffusion in itself. The only thing you can say is that the pair X and L are a bivariant diffusion.

If you bring this L over to the side, you can see the coefficients for DL depend on L and X-- and same thing with the volatility part. So anyway, we place the structure.

And the idea is that we know, from looking at Bloomberg, what the risk neutral drift of X is-- that's BQX. We know that function. We know what the diffusion coefficient of x is. That's the function A of X.

We know what the risk neutral drift of L is-- that's that function R of X. But we don't know the volatility of John Long's numeraire portfolio. That's the function sigma L of X. And if only we could find it, we would actually know how to determine the real world drift. And remember I was saying if IBM and you could put an American bank account, and let's say there was certain growth rate there. And then if instead you were putting those gains in a British bank account, you'd achieve a different growth rate. And I was stressing that the correlation of dollar pound with IBM was important for determining that growth rate. And I stand by that.

When you're in a one factor world, that correlation can only be one. And so that's what's happening here. We're in a one factor world, and that correlation is one. And the other thing that affects the growth rate, though, of your British bank account balance is actually the volatility exchange rate. So what actually matters is the covariant between the British exchange rate and IBM. That covariance depends on both the correlation and the volatility of the FX rate. So you can think of the FX rate as here John Long's numeraire portfolio. And so that sigma L is sort of the key. It's like we've set things up so we know the correlation, but we still don't know the covariance. And that's what's actually relevant. So as soon as we get the sigma L, we'll know the covariance. So we'll be in shape.

So we got to find that volatility function sigma L. And now I know many of you have classes, so I'm going to have to start moving.

AUDIENCE: Now, Peter, people will have access to these slides afterwards. And so, I'm just seeing you've got another 15 slides left.

PETER CARR: Yes, well actually, you'll be glad to know that five of those are disclaimers. If I could move along--

AUDIENCE: But the point is to what--

PETER CARR: The key is towards the end. Yes, absolutely. We're very close. OK. So I'll be done in two minutes. So basically, where we are now is we're going to make one more assumption that the value of John Long's numeraire portfolio is a function of X and D. OK then, let's say we've made all our assumptions. And where it goes is that the assumptions imply that this value function splits into an unknown positive function of x, and an unknown positive function of time. And when you kind of further analyze, you find that the unknown function of time is an exponential function of time. And the unknown function of x solves an ordinary differential equation of this kind.

So this is called a Sturm Liouville problem. And it turns out that Sturm Liouville were the only mathematicians I've mentioned in this talk who were not at MIT. And they actually solved this problem. And one of the things they show is that when you're searching for functions pi and scalar's lambda that solve this problem, there's only one solution that delivers you a positive function pi. And so this is how you get uniqueness. Remember I was saying back with 11, so we're searching for like a 10 vector and a scalar. Now the 10 vector is a function. And that function is pi, and the scalar's lambda.

So the point is is that the math implies there's a unique solution to the problem. So we learn the volatility of the numeraire portfolio in the end. And then we learn the drifts of everything you want to know under the market's beliefs. So that's the gist of it. So then there's been work on trying to extend to unbounded intervals. And basically, in the famous Black Scholes model, this effort fails, whereas in the less famous but still important Cox, Ingersoll, Ross model, this effort succeeds. So the sort of punchline is that when it comes to unbounded state space, the theory's open. So if there's a grad student in the room who wants a good dissertation problem, this is it. OK. So that's all I wanted to say today. Thanks.