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PROFESSOR: All right, OK, let's get started. So before I make introduction, let me just make a few announcements. A few of you came to us asking about the grading for the term. And some feel the problem sets may be on the difficult side, and some of you haven't done all of them, and some of you have done more.

So we just want to let you know that the most important thing to us in grading is really you show your effort in terms of learning. And we purposely made the problem sets more difficult than the lecture, so you can-- if you want to dig in deeper so you have the opportunity to know more. But by no means we expect you to finish or feel easy in solving all the problem sets.

So I just want to put you at ease that if that's your concern, that's definitely-- you don't need to worry about it. And we will be really just evaluating your effort. And based on what do we observed so far, we actually believe every single one of you is doing quite well.

So you shouldn't worry about your performance at the class. So continue to do a good job on your class participation, and do some of the problem sets. And then you will be in fine shape for your grade. So that's all of that.

So without any further delay, let me introduce my colleague, Doctor Stephen Blythe. I'll be very brief. And he's-- Stephen is doing two jobs at the same time. He's responsible for the all the public markets at Harvard Management, as well as being a professor of practice at Harvard. So with that, I turn to Stephen.

STEPHEN BLYTHE: OK, well, thank you, and thank you for having me to speak this afternoon. Before I begin, I wanted to ask you a question. So I'm speaking, actually, at almost exactly the 20th anniversary of something very important.

So on the 19th of October, 1993, which I guess might be the birthday of some of you, but almost exactly 20 years ago, Congress voted 264 to 159-- I actually remember the count of the vote-- to do something. So anybody like to guess what they voted to do on the 19th of October, 1993 that might be tangentially relevant to finance and quantitative finance? Anyone here from HMC is not allowed to answer. Anybody-- any guesses? Any ideas at all coming to people's minds?

AUDIENCE: Was it Gramm-Leach-Bliley?

STEPHEN No.

BLYTHE:

AUDIENCE: [INAUDIBLE]?

STEPHEN No, but good guess. But actually, that is actually too related to finance, actually. This

BLYTHE: is actually-- wasn't actually directly financially related, so that was related to [INAUDIBLE]. Anybody else think about? What does Congress usually do?

AUDIENCE: [INAUDIBLE].

[LAUGHTER]

STEPHEN No, no ideas whatsoever? What do you think Congress did 20 years ago? They
BLYTHE: voted to do something.

OK, well, what Congress do usually is they cut money for something. So they voted to cut financing to something. So what did they cut financing to? Anybody guess?

I know this isn't business school. In business school, it would be, like, right, you're failed. No class participation-- you failed. You've got to say something in business school.

So I know it's not business school. But anyway-- and I don't teach in business school, either. But this is actually-- these round desks make me think of business school and striding into the middle of the room, and saying OK, come on.

Fortunately, I don't have names, otherwise I'd pick on you.

No, no guesses-- no guesses whatsoever? I've got to take this up the road to Harvard Square, and say I've taught at MIT. No one had any guesses with this question-- one guess, actually, the gentleman here. What did they cancel the financing for in 1993?

I'll say it was the Superconducting Super Collider underneath Texas just south of Dallas. So \$2 billion had been spent on the Super Collider And the budget had expanded from, I think, \$6 to \$11 billion. So they, by canceling, had a \$9 billion dollar savings. This is 20 years ago-- almost exactly.

And as a result of that-- one result of that-- was, of course, the academic job market for physicists collapsed overnight. And two of my roommates were theoretical physicists at Harvard. And they basically realised their job prospects in academia had vaporized instantaneously that day.

And both of them, within six months, had found jobs with Goldman Sachs in New York. And they catalyzed they-- they and the cohort-- they're called the Superconducting Super Collider generation. If you ever wondered why people like myself and like Jake got PhDs in quantitative subjects around the turn of-- around 1990 to 1993-- all ended up in a financial path, part of it is due to Congress cancelling the Superconducting Super Collider.

Because this cohort catalyzed this growth in quantitative finance. Actually, they created a field-- financial engineering-- which you are all somewhat interested in by taking this class. And they also created a career path-- quantitative analyst, or quant, which really did not exist before 1993.

And that growth of mathematical finance, financial engineering, quantitative finance-- however you want to look at it-- was basically exponential from 1993 up until 2008 and the financial crisis exactly five years ago, funnily enough. And since then, it's been a little bit rockier. So if you're actually interested in this aftermath of the physics funding-- what's interesting is the Large Hadron Collider, which you might

know is up and running in Geneva and just found the Higgs Boson, has sort of reversed the trend somewhat.

So there used to be a whole cohort of people going into finance instead of physics. Now, because finance has this somewhat pejorative nature to it-- people don't like bankers generally, and they kind of like physicists who find the Higgs Boson and get a Nobel Prize-- maybe we're getting reversal. But anyway, we're still in finance.

I've, as Jay mentioned, well, I did mention, I was originally in academics. I was actually a mathematics faculty member in London when I got my PhD I got my PhD from Harvard.

And in 1993, I was an academic. And all my friends-- I saw them go to finance. So I followed them, spent a career in New York, and then came back to Harvard in 2006 to run a part of the endowment.

And I started teaching. So just as a plug-- for those of you interested in mathematical finance and applications of mathematics finance, I teach a course at Harvard. It's an upper level undergraduate course called Applied Quantitative Finance, which of course you can cross-register for.

And today is also the one-week anniversary of the publication of my book. So if you're interested in what my course is about, you can just buy my book. It's only \$30. And I'll sign it. It's first edition, first printing, first impression book, *Introduction to Quantitative Finance*.

And that is what the course is. It's quite distilled. When this book came out, I thought, that's really thin. This is three years of my life's work. It's come out-- it's very thin.

But I like to think it's like whiskey. It's well distilled, and highly potent, and you have to sip it, and take it bit by bit. Anyway, that is the book of my class.

And the genesis of the class was really that, when I've been on Wall Street, and I was a colleague of Jake's at Morgan Stanley in this rapidly growing quantitative

finance field, we encountered on the trading desk in the late 1990s and the early 2000s problems from the real economy-- things that we had to trade. We were-- things that were coming to us on the trading desk that required subtle understanding of the underlying theory. So that we, in essence, we built theoretical framework to solve the problems that were given to us by the financial markets.

So that period, especially around the turn of the century, there's a big growth in derivatives markets-- which options, futures, forwards, et cetera, swaps. And we needed to build theoretical tools to tackle them. And that's really what the course was evolved out of, to build the appropriate theoretical framework, motivated by the problems we encountered.

Why I enthuse about the subject-- and I really like teaching the subject-- is that there is an impression that qualitative finance is a very arcane and contrived subject-- just a whole bunch of PhDs on Wall Street coming up with crazy ideas. And they need complicated mathematics that's just complicated for the sake of complexity. And the theory is just sort of a contrived theory.

But in fact, at the heart of Wall Street is that the real economy demands some of these products by supply and demand. There are actual, real participants in the financial markets who want to trade derivatives. And therefore, in order to understand them, you need to develop a theory. So it's actually driven by real examples.

That's one part. The other part is that the theory that comes out of it, and in particular the approach I take here, I think is just very elegant. OK, so there's some subtlety and elegance and beauty to the underlying theory that comes out of addressing real problems.

This course, and the way that I teach finance, is very probability centric. You probably realize from the lectures you've seen already in this class, there are many different approaches, many different methods that are used in finance-- stochastic calculus, partial differential equations, simulation, and so on. The classical derivation of Black-Scholes is, well, it's the solution of the PDE.

OK, that has appealed to people. In fact, this is why in some ways, quantitative finance is a broad church, because whether you're a physicist, or probabilist, or a chemical engineer, all the techniques you learn can be applied. You know stochastic calculus. You know differential equations. They can be applied.

But the path that I take in this class is very much through the probabilistic route, which is my background as a probabilist, as an academic, or a statistician as an academic. And this is, in particular, I think, a very elegant path to understand finance, and the linkage between derivative products-- which might seem contrived-- and probability distributions, which is sort of natural things for probabilists.

So this, what we're going to talk about today, is really this link, which I call option probability duality. Which, in essence, in the simplest form, is just saying, option prices-- they're just probability distributions. Therefore, these complicated derivatives that people talk about-- all these options, these financial engineers, these quantities exotics-- we're really just talking about probability distributions.

We can go between them-- option prices, probabilities, and distributions-- back and forth in a very elegant way. What I love about this subject in particular is that to get to that point where we see this duality does not need a whole framework and infrastructure of complicated definitions, or formulae, or option pricing formulae, or so on. So that's what I'm going to try and do in this hour or so, is introduce this concept of option price, probability duality. And show how the natural-- so there's a natural duality that can be seen in a number of different ways.

OK, so we're going to need a few definitions that should be familiar to you. We're going to define three assets. We have a call option, which we know about, a zero coupon bond-- called a zed cee bee. This is the one thing I haven't become Americanized on. I still call this zed. It's a-- other things I've become-- and then a digital option.

OK, all right, so what are they? Well, they're all going to be defined by their payouts at maturity. OK, so we're going to have some maturity capital, T , and some

underlying asset, S , the stock, with some price, S_T . OK, so we know that the call option has payout at T . So that's called payout at T . So T is some fixed time in the future. We will change in the future to some fixed time. This is simply the max of S_T minus K and 0 . That's a call option.

You can go through the right to buy, et cetera, et cetera. But it's clear it's just value at maturity is just the max of S_T minus K and 0 . The zero coupon bond with maturity T is just something that's worth 1 at time T . So that's just part one. That's definition-- so you can think of these all as definitions.

And then the digital option is just the indicator function of S_T being greater than K . So here, T is the maturity. K is the strike. So T maturity, K is strike. And these are three assets.

So this is, in some sense, the payout function. All derivative products can be defined in terms of a function-- not all of them. Many derivative products can be defined just as a function of S_T . And here are three functions of S_T . [INAUDIBLE]

And then I'm just going to get notation for the price at t less than or equal to T . We can think about little t as current time today, or we can think of some future time between now and capital T . I'm just going to introduce notation. Every different finance book uses different notation, so just C for call price, with strike K at little t with maturity big T . OK, just that notation.

The zero coupon bond-- the price at little t -- let's call that Z . That's the price of little t . And the digital-- we'll just call that D .

So this is what we're going to set this up. Actually, you could have a whole lecture on why notation-- different notation. K and capital T are actually embedded in the terms of the contract. Little t is in my calendar time.

So you might think why don't you put K and capital T somewhere else? Well, when you get actually to modeling derivatives, you like to be moving both maturity and [INAUDIBLE] time and calendar time. That's why I just write it like that. But there's no-- so $C_{K, t, T}$ is the price at time of little t of a call with maturity

capital T and strike K .

Black-Scholes and other option pricing formula are all about determining this-- for t less than T . Because clearly we know that the price at maturity is simply the payout. I mean, that's, again, just the definition.

So that's trivial. But we want to find out what the price is at little t . So that's the whole path of finance-- Black-Scholes and other option pricing methodology is working out this. But we're actually going to go down a different route.

So what we're going to do-- we're going to construct a portfolio. So consider as a portfolio of what? We're going to consist of two calls. OK, we're going to have λ calls with strike K . OK, so this is the amount holding. And everything is going to be with maturity, capital T .

So λ calls with strike K , and maturity T , and minus λ calls with strike $K + 1/\lambda$. We'll just consider that portfolio-- consists of two options. All right, well, this price at T -- that's easy. We just right it in terms of λ times the price of the call with strike K , minus λ call with strike $K + 1/\lambda$ -- just by definition. This is price at T .

OK, well, let's look at its payout at time capital T graphically. So we know about call options. The payout function is just the hockey stick shape, clearly.

That's confusing to people from the UK, because in the UK, hockey means field hockey, not ice hockey. And of course, the hockey stick shape in field hockey looks very different. Anyway, that's-- you understand what the path of a call is.

Clearly, this payout function of a call looks like this. Well, putting this payout of λ calls of strike K minus λ calls of strike $K + 1/\lambda$ -- let's assume λ is positive for the time being. What's it look like?

Well, 0 below K is flat above $K + 1/\lambda$. It has slope λ , and has value 1 here. You should be able to see that easily.

So that's the payout. This is called call spread-- just the spread between two calls, and has this payout function. OK, so a natural thing to do here, it being a mathematics class, let's take limits. Just let λ tend to infinity.

Well, then, this becomes the partial derivative of the call price with respect to K , or the negative of it. So this tends to minus. OK, let's just-- so that's that.

And then this, of course as λ goes to infinity, this stays at 1. So this tends to payout function that looks like that. OK? This is easy calculus. This is just by inspection. OK, so this, clearly, is the payout of the digital.

Of the-- strictly a digital call, but that's called the digital option. Just as a note, here it's, just greater than. You might think, OK, it doesn't matter if it's greater than or equal to.

Well, in practice, the chance of something equalling a number exactly is 0-- I mean, if it's a continuous distribution. In theory, I should say, the chance of something actually nailing the strike, actually being equal to K , is 0, so it doesn't really matter whether you define this as greater than, or greater than or equal to. But in practice, of course, finance is in discrete time, because you don't quote things to a million decimal places. So certain assets, actually, which are quoted only in eighths or 16ths or 32nds or 64ths, this matters, actually, whether it's defined as greater than or greater than or equal to. But theoretically, it doesn't make any difference.

OK, so we've got the call spread tending to the digital. All right, so this tends to-- so the limit of this call spread-- of this price of the call spread-- is the digital. And so we know that because this is the price at t . This is the payout at capital T . The price of the digital must equal just the partial derivative with respect to strike of the call price.

So that's just a nice, little result. Where does this bring in probability? So this is the next.

OK, so this is where we'll make one assumption. And it's actually a very important and fundamental assumption. And it's fundamental because it's called The Fundamental Theorem of Finance, or the Fundamental Theorem of Asset Prices.

So I call this FTAP-- Fundamental Theorem of Asset Pricing.

By this theorem, which we are going to assume here, the intuitive answer is correct, meaning that prices today are expected values. There's the expectation of a future payout. So by FTAP, the price at t is expected payout at time capital T , suitably discounted.

So there's both something very straightforward here, and something very deep. If you think about how much would you pay for a contract that gives you \$1 if an event happens-- in this case, the event being stock being greater than K at maturity. You would intuitively think that's related to the probability of the event happening.

How much will you pay to see the dollar if a coin comes up heads? You'd pay a half, probably. It's very, very intuitive.

But the deepness is, this actually holds under a particular probability distribution. I'm not going to go into that here, but by the fundamental theorem, this is true. So I can write, in the case of the digital, the digital price equals the discounted-- and we'll explain why we want to put the zero coupon bond price here-- that's the present value of a dollar at time t . It's just a discount factor. It's very trivial, but it's written in terms of an asset price-- times the expected value of the payout.

So either you take this as this makes a lot of sense-- the discounted expected payout-- or you can say, I don't understand this. I want to find out about the Fundamental Theorem Asset Pricing, which we will prove in my class. But this intuitively makes sense. The key here is that the expected value actually has to be taken out under the appropriate distribution, called the risk neutral distribution.

But this formula holds-- in fact, strictly. I'll write this is just for-- what holds is the price at time little t divided by zero coupon bond is a martingale-- for those of you into probability theory. This gets probabilists very excited, of course, because they love martingales. Everyone in probability theory loves martingales-- lot of theorems about martingales.

And you'll see, of course that this is actually a restatement of this assertion.

Because Z , capital D , capital T is 1. So this statement here is simply a re-expression of this martingale condition.

So I'll just pause here. Just from a probability point of view, when I learned probability, it was under David Williams, who wrote the book *Probability With Martingales*, which is a wonderful book. And I thought martingale is a great thing.

So I was sort of happy. It took me about seven or eight years of being in finance to realize there are a whole lot of martingales floating around. Because this actual approach-- this formalization of asset pricing really only became embraced on the trade floor around the early 2000's, even though the underlying theory was always there-- this idea of these martingales. Anyway, so this is-- and this, of course, is simply-- the expected value of the indicator function is just the probability of the event. OK.

All right, so now I've won by intuition. Just here's the probability of the payout occurring at price digital. I've also priced the digital by taking the limit of call spreads. So now I'm just going to equate them.

So by equating these two prices for the digital, I simply get that the derivative of the call price with respect to strike equals the discounted probability of the stock being above K . I've just reorganized a little bit, take 1 minus. So I get the probability that-- well, I can clearly reorganize again and get-- all right, so if I want to simply get the cumulative distribution function, it's just 1 minus this. So divide here, take 1 minus.

OK, so I get the cumulative distribution function for the stock price at T is equal to 1 plus the dC by dK times 1 over Z . I'm just rearranging. So here now is the cumulative distribution function.

Clearly, I just need to differentiate again to get the probability density function. So here's where the notation gets kind of messy, but clearly the probability density function of-- f for my random variable S sub T -- so the density of-- express that as-- I always-- probabilists, whenever they talk about densities, they always want to say f of x . And it's the same with me.

That's f of x . Here's the density is simply just the next, the second derivative. We'll take the derivative of this. It's the second derivative of the call price with respect to strike, evaluated at little x .

All right, so what we've done here is start off with simple definition of three assets, price to digital in two different ways. And now we have a rather elegant linkage between call prices-- C -- and the density of the random variable that is the underlying stock price at capital T . OK, so we've established one side of the duality, which is given the set of call prices for all K , I can then uniquely determine the density of the underlying asset.

So you might think, OK, this is kind of nice. How does this actually work in practice? Do we actually think in terms of probability trading? We just said that call options are equivalent to probability density functions.

Well, actually, there's a very neat way of accessing this density function through another portfolio of options. OK, so this is actually where we get-- to me it's the practical relevance of some of this theory. So let me just show you that.

So we're going to consider another portfolio. So here we consider portfolio as follows-- it's actually going to be the difference between two call spreads. So λ calls with strike $K - 1$ over λ . Minus 2λ calls with strike K , and λ calls with strike $K + 1$ over λ -- again, λ positive.

OK, why are we doing this? Let's just stop for a bit of intuition. Here we see in the call spread the discrete approximation to the first derivative of call price with respect to strike. So clearly, if I want to approximate the second derivative, I'm going to take the difference between two call spreads appropriately scaled.

You're now going to have to have a little-- there's got to be another λ coming in here at some point. This is just the difference between two call spreads, so that's the difference between two approximations of the first derivative. So I'm going to have to scale by λ in order to approximate the second derivative.

So this is actually called a call butterfly. And this is a beautiful thing for two reasons. One is they actually trade a lot-- surprisingly. This is not a contrived thing I just made up.

A, it trades a lot, so you can actually trade this thing. The second is you can kind of imagine the right scaling of this call butterfly is going to approximate the second derivative, and that's approximating the density function. So this is a traded object that will approximate the density function. Yeah, you have a question?

AUDIENCE: Yeah, I have a question. In the real world, you cannot really [INAUDIBLE] infinitely small, so they have some [INAUDIBLE] way how to approximate that?

STEPHEN BLYTHE: Yeah, so that's a good point. Yeah, so the question is how, in practice, we can't go infinitely small, which is true. But we can go pretty small.

So in interest rates, we might be able to trade a 150, 160, 170 call butterfly or equivalent-- 10 basis points wide. That's a-- it's a reasonable approximation to the probability of being in that interval. So these are all, I mean, you make a good point.

In fact, all of finance is discrete, in my view. So continuous-time finance is done in continuous time because the theory is much more elegant. But in practice, it's discrete in time and space. You can only trade finitely often in a day, and so on.

I won't going into the detail, but you can see the price. Let me just write down the first. The price of this I have just expressed as the difference between two call spreads. So it's λ times the call spread from $1 - \lambda$ to K , so $K, 1 - \lambda$ to K , minus the call spread from K to $K + 1/\lambda$.

OK, so the difference between two call spreads-- we'll call this-- this is the butterfly. We're just going to use temporary notation, call that B , B for butterfly. So the price B , and then you get confused. It's B centered at K with width λ . No one ever uses this notation outside this one section of my class, so that's why, but it's just handy for this.

So that is the butterfly price is equal to the difference in these two call spreads.

What I want to do is, I want to take limits of this, suitably scaled, to get the second derivative. And if you just take λ times BK of $\lambda \Delta t$ is indeed, approximately-- if I take limits is exactly-- the second derivative of call price.

OK, so here's how I'm accessing the second derivative through a portfolio of traded options. All right? And so the price of this butterfly, B , if I just reorganize and substitute-- so I get BK -- for large λ , i.e. a small interval-- is approximately 1 over λ times the density function-- actually, evaluated at K . So I have obtained this density function by this traded portfolio.

And to your point about we're not getting infinitely small. That's absolutely right. But if you think about what the density-- when you learn about density functions for the first time, you say the density function at x times a small interval is the probability of being in that small interval.

All right, so when we think about the density function f of x , if you have a small interval of Δx , then clearly the probability of being in this interval is approximately f of x , Δx . In the limit, that is true. So what we're showing here, if you actually think about what interval we're looking at, we're actually looking at in this call butterfly-- if you were actually to draw it out, this call butterfly looks like that around K .

It actually-- it's a little triangle. It's not actually a rectangle, but it's a little triangle of width 2 over λ . OK, so it is actually-- this is the area of this triangle-- 2 over λ times $1/2$ times f of x . And that's actually this, right? So this has width 2 over λ .

OK, so in fact, we've got here exactly an approximate-- exactly approximation-- that doesn't sound right. But it's entirely analogous to the approximation of the probability of being a small interval. Here is the probability of being in this interval here-- just the area under that is exactly 1 over λ f of x .

So here is actually something that people do do, is they say, OK, I will look at the price of this butterfly, which gives me the probability of this underlying random

variable ending up around K . I'll make a judgment whether I agree with that probability or not. And if I think that probability is higher than this price implies, then I'll do a trade. I'll buy it. I'll buy that butterfly.

So there is actually an active market in butterflies, and so I think an active trading in probabilities-- probabilities of the underlying variable being at K at maturity. So OK, so that's the first linkage here. Both-- the density is the second derivative, and the second derivative is essentially a portfolio of traded options.

And none of this is dependent on the actual price of the call option, in the sense that this holds regardless. Clearly, this is a function of the price of the call option, but I don't need any model for the option price to hold, in order for these relationships to hold. So these are model-independent relationships, clearly.

If you were to put the Black-Scholes formula into C -- Black-Scholes formula of the call price-- and take the second derivative with respect to K , which would be a mess, you'll end up with a log-normal distribution. Because that's what actually the Black-Scholes formula is, is expected value of the payout under a log-normal distribution. And that will hold. So this will hold for that.

AUDIENCE: [INAUDIBLE]?

STEPHEN Yes.

BLYTHE:

AUDIENCE: The last [INAUDIBLE] depends on the [INAUDIBLE].

STEPHEN Yes, it does.

BLYTHE:

AUDIENCE: But the right-hand side does not. What's the role of that?

STEPHEN Yeah, that's a really good point. I've been loose in my notation. So here what is it?

BLYTHE: It's actually the conditional distribution of S capital T , given S little t . So this is the conditional distribution, given that we're currently at time little t with price S little t for the distribution at time capital T . So that's where it comes in.

That's absolutely right. So in fact, this expected value strictly should be conditional on \mathcal{F}_t . This probability is a probability conditional on \mathcal{F}_t -- absolutely. And in fact, this martingale condition is the martingales with respect to \mathcal{F}_t . So that's where the little t comes in.

AUDIENCE: [INAUDIBLE]?

STEPHEN Here, yes, sorry. That's $1/Z$. So it's just a constant here.

BLYTHE:

This number, especially because interest rates are so low in US, so this number is so close to 1 that you always forget about that. Not when we're trading, but when you, oh well, this is just a-- if you just think about which one is-- this is a quantity that's in the future. It's call prices, so that's how you kind of go.

All right, so that's the first bit. So when I was an undergraduate, actually, learning probability, one thing I learned about probability was from my probability lecturer, who said, the attention span of students is no more than about 40 minutes. So there's no point lecturing continuously for 40 minutes, because people will just start switching off after 40 minutes.

So rather than wait, just have a break and waste the time, the lecturer said, I'm just going to give you some random information in the break, and then we'll go back to probability. So I learned that from 25 years ago. I can't remember-- I actually remember the material. I can't remember any of the random material.

So that's what I do in my lectures, is I break them up, and talk about something random. So I thought I'd do that here as well, with some-- not completely random. So this is somewhat applicable, this being a math class.

So how many of you are math concentrators or applied math concentrators? One, two-- a lot, math concentrators-- especially for the applied math concentrators, going straight to the conclusion-- your entire syllabus was generated at Cambridge University. That's the conclusion.

So anyway, here's the story. So back in the 19th century, the Cambridge Mathematics degree-- the undergraduate Mathematics degree-- was the most prestigious degree in the world. In fact, it was actually the first undergraduate degree with a written examination was Cambridge Mathematics. So they have a lot to be responsible for.

And each year, people took the exam. And they were ranked. And that ranking was published in the *Times* of London-- so the national newspaper. And the people who got first-class degrees-- so summa degrees-- in Cambridge Mathematics were called wranglers, and still are called wranglers, actually.

And the reason they're called wranglers was from way back in the 17th century where, before they had exams-- or 18th century, I should say, before they had exams-- instead of writing down exam, you have to argue, or dispute, or wrangle with your professor to get to pass the class. So that's where wrangler comes from. So these people who got the first-class degree are called wranglers, and they're ranked.

And basically, the senior wrangler was a very famous person in the UK in the 19th century. And a lot of them turned out to be quite successful. So here are a few wranglers. I've just got this one-- I can't reach that, but [INAUDIBLE].

So some of you might recognize-- and I just want to tell you a quick story about one of them. OK, so let's start 1841, senior wrangler was George Stokes-- so basic fluid dynamics-- the whole of fluid dynamics-- that's George Stokes. 1854, second wrangler-- this is-- who was the first wrangler? The second wrangler was James Maxwell, so electrodynamics, Maxwell equations. He was the second. And I can't quite work out who was the first.

1880, the second wrangler was J.J. Thompson, so electrons, atomic physics, that comes out of-- he was only second. 1865, senior wrangler Lord Raleigh. So he was the sky is blue. He was first.

So they're a pretty good bunch. So the story-- the best of 1845-- I'm going back--

the second wrangler was Lord Kelvin, so absolute zero, amongst other things, of course. But absolute zero-- he was second wrangler.

And the great story about him, he was the most talented mathematician of his-- of the decade. And he was such a lock for senior wrangler that-- and I actually read the biography, so this is a sort of true statement-- that he sent his servant to the Senate House where these things are being read out, and with a request, "Tell me who is second wrangler." And the servant came back, and said, you, sir.

And because he was such a lock to be first wrangler. And in fact, what happened was a question on the mathematical exam was a theorem that he had proved two years before in the Cambridge Mathematical Journal. So his theorem was set on the exam.

Because he had not memorized it, so he had to reprove it, whereas the person who became senior wrangler had memorized the proof, and was able to regenerate it. In those days, there was a lot of cramming to be done in these exams. So the guy who-- Stephen Parkinson was senior wrangler. He went on to be FRS, and eminent. But he wasn't-- so anyway, so here's the applied math syllabus.

Here's a couple of other ones which I really like. In 1904, John Maynard Keynes was at 12th wrangler. Now, I can tell the story either way, depending on whether I'm in an audience of economists, or an audience of mathematicians. Since I'm in an audience of mathematicians, I like to say the greatest economist was so poor at mathematics, he only managed to be 12th wrangler.

There are 11 better mathematicians in the UK in that year. So he was obviously not that great. If I was talking to economists, I would say, this guy is so brilliant that his main field was economics, and yet as part time, he's able to be the 12th best mathematician in the UK.

So last one I wanted to talk about-- 1879-- here's a quiz. This one you have to have some answers for. OK, so 1980 something-- I can't remember what it is-- so here's one, here's two, here's three. I'm going to give you one and two. You've got to fill in

three.

You probably aren't going to be able to get this one yet, but this is-- Andrew Alan, senior wrangler, George Walker, second wrangler, and number three is the question. That's the question-- 1980, Hakeem Olajuwon, Sam Bowie, question mark-- who's question mark? Do we know which sport these people play?

AUDIENCE: That one's Michael Jordan.

STEPHEN BLYTHE: Yes, right. There we go, that's Michael Jordan-- exactly. This question could go on forever in the UK because they don't-- so Michael Jordan, famously, was only picked third in the NBA draft in 1984, was that-- four or five, something like that. So Hakeem Olajuwon was actually pretty good, but Sam Bowie was a total bust. But he was third.

So in 1879, in the Cambridge Mathematics Tripos, these two people you never heard of, who were first and second. And the person who came third, you've probably heard of him. This is more of a statistics thing. People know about correlation? What's the correlation-- who's the correlation coefficient named after?

AUDIENCE: Pearson.

STEPHEN BLYTHE: Pearson, you've got Karl Pearson. So Karl Pearson was the third wrangler in 1879. And the founder of statistics-- he founded the first ever Statistics Department, and obviously invented correlation with Gould-- Gould and Pearson.

Anyway, he was only the third wrangler. And unfortunately, these people have very common names, so I have no idea what they went on to do. To Google these people is not very effective. Anyway, so that's the story of Cambridge Mathematics-- lots of good stuff.

All right, so in the last half hour, I just want to go the other way from-- so the other way-- we went from option prices to probability. Let's go from probability to option price. We sort of already have, actually. This is what the Fundamental Theorem does.

If we're thinking-- if we take on trust that the Fundamental Theorem holds, namely option prices today are the discounted expected payout at maturity. OK, let's take that on trust. Then we're going from probability distribution to option price in the following way.

So let's actually state the Fundamental Theorem, FTAP. OK, so I'm going to go general derivative D is-- D , digital D , derivative. It's-- so derivative with payout at T .

So this could be the digital payout. It could be call option payout. It could be one. And price-- OK, so often, we actually think about payout function as just a simple function of the stock price.

But this notation is useful when we think about the price as being martingales. Then what is FTAP? D -- the ratio of the price to the zero coupon bond is equal to-- is a martingale. In other words, its expected value under the special distribution, risk-neutral distribution, of the payout at maturity.

And to you point, it's conditional on \mathcal{S}_t . So this is the proper statement. So this is the Fundamental Theorem of Asset Prices. In words, it's saying this ratio is a martingale with respect to the stock price under the risk-neutral distribution.

That's the statement of the Fundamental Theorem. This is actually rather neat to prove in the binomial tree, two-state world. It's very, very difficult to prove in continuous time. This is Harrison and Kreps in 1979.

It's the proof that, however many times you look at it, you're only probably going to get through two or three pages before thinking, OK, that's hard. But it was done. So this is, you can imagine continuous time, infinite amount of trading, infinite states of the world.

OK, so now this, of course, is 1. And this can come up. These are known at time little t . So if I'm thinking at-- if I'm at current time little t , therefore, the derivative price is what we had before-- the expected payout.

OK, so this is rather a nice expression. And now we can actually just write down

what this. This is the expected value of a function of a random variable.

So this is just the integral of G of x , f of x , dx , where this is the density of the random variable at time capital D , conditional on being at St . So this is conditional at St . So this is a restatement of the Fundamental Theorem. So this is essentially the Fundamental Theorem. And this is an intuition made good, or intuition made real-- expected payouts.

This is sometimes called LOTUS-- the lure of the unconscious statistician. Just the expected value of g of X is $\int g$ of x , f of x . That's not immediate from the definition of expected value. You should really work out the density of g of x . And then $\int g$ of x -- the density of g of x dx , but it turns out to be this. So that's a very nice, nice result.

OK, so here is now a way of going from density to price. If I put in the call option payout for g , and I have the density, I can then derive the price C . So if you like, the way I go from density to probability distribution to option price is exactly the Fundamental Theorem. The route I take is the Fundamental Theorem.

OK, so FTAP, the Fundamental Theorem of Asset Pricing, means I can go from the probability density to the price of a derivative for any derivative. All right, OK, so now we can go either way-- density to derivative or call price to density. You might say, hang on a sec. We've only gone from-- we need the call prices to get the density.

Well, of course, we can go via an intermediate step. So to get from the call price to an arbitrary derivative price, I just go via the density. So in particular-- this is restating-- knowledge of all the call prices for all K determines this derivative payout for any g .

So if I know all calls, I know the density. And then if I know the density, I know an arbitrary derivative price. It's obvious as we stated here. But what this is saying is, the call options often are introduced as this-- why are they important-- are the spanning set of all derivative prices.

So calls span-- call prices span all derivative prices. And they are a particular type of derivative-- the ones that are determined exactly by their payout at maturity. One can imagine other things that are a function of the path or whatever. But this is a particular derivative price. European derivative prices are determined by calls.

OK, so that's kind of nice-- sort of obvious, elegant. There's two other ways of looking at this, though. If I think about my function g -- so consider function g -- OK, so that determines my derivative. So it determines, defines the derivative by its payout at maturity.

Let's just graph it. OK, so it might look-- let's just assume first it's piecewise linear, so it looks like-- so suppose this is g . Well you can kind of see I can replicate this portfolio, or this option, by a portfolio of calls-- in fact, a linear combination of calls.

Right, I have no cause but to say K_1 this is K_2, K_3, K_4, K_5 , et cetera. You can see what the portfolio of calls will be in order to replicate this payout at maturity. There'll be a certain amount of calls with strike K_1 , so that the slope is right, minus a certain amount of K_2 to get this slope, plus a certain amount of K_3 , minus K_4 , minus K_5 , plus K_6 , et cetera. So, in this case, if the piecewise linear g , replicating portfolio of calls, it's obvious. So if I can replicate the payout exactly at maturity, the price at time little t of this derivative must be the price at little t of the replicating portfolio.

That's actually a-- I'll do that early on in my class, and of the 100 people, everyone says, OK, that makes sense. And someone says, does that always have to be the case? And it's actually a really, really good question. Here, I was about to just hand wave over it.

Is it the case that if I have one derivative contract with this payout at maturity, and I have a linear combination of calls with the identical payout at maturity, capital T , must these two portfolios have the same value at little t ? Well, one would think so, because they're both the same of maturity, so they must both be the same thing. They're just constructed differently.

And the assumption of no arbitrage-- which underpins everything, in some sense,

what we're doing-- would allow you to say yes, indeed, that is true. And in fact, it's actually the fundamental of finance, right? If two things are worth a dollar in a year's time, they're going to be worth the same today. That's what we're saying.

If you can match the portfolio at t , that is actually the definition of-- it follows immediately from no arbitrage. What has been interesting in finance, especially since 2008, is that that-- this assumption-- has broken down. In other words, I can hold a portfolio of things when aggregated have exactly this payout, against an option with exactly this payout, and be paid for that.

And this is actually really-- it's a very fascinating thing, to think about actually, the dynamics of financial markets when arbitrage can break down. What is the main theme here is that when capital T is a long way in the future-- 10 years, 20 years-- there's nothing to stop the price of the option in the replicating portfolio going arbitrarily wide, other than people believing that it has to be equal.

The only way you can guarantee the two things to be equal is by holding it until capital T -- for 10 years, 20 years. In the meantime, those prices can move. Empirically, they've been shown to move away from each other.

So there's actually a deep economic question here. So if there is the presence of arbitrage in the markets, then arbitrage can be arbitrarily big. Because you're saying there aren't enough. There's not enough capital, or that's not enough risk capital, for people to come in and say, OK, these two things have to be worth the same in 10 year's time. Therefore, I'm prepared to buy one \$1 cheaper than the other.

It's actually a question really relevant to the Harvard endowment. We're a long-term investor. You say, why doesn't the Harvard endowment, if these two things are \$1 apart, buy the things that's \$1 cheaper, and just hold them 10 years, make the dollar?

Well, we'd like to, but if we think they're going to be \$1 apart, and they're going to go to \$10 apart, we don't want to buy them at \$1 apart. We want to buy them at \$10 apart. I mean, yes, we're a long term investor, but we care about our annual

returns, or five-year returns.

Suppose this is a 20-year trade. This is very prevalent when these things are 20 years out. Anyway, it's a whole-- this is-- it's a little bit-- it's a foundational issue. It's this thing where it could shake the foundational underpinnings of quantitative finance if you don't allow this replicating portfolio to have the same price as the actual option.

But mathematically, you can see you can replicate it, certainly capital T , and therefore the price at time little t is just the linear combination of call prices. OK, so let's assume that. And then obviously, continuous function can be arbitrarily well approximated by piecewise linear function.

Therefore, any function at time-- any function of this form-- a derivative when compared to that form can be replicated by a portfolio of call options. So we can sort of hand wave to kind of say, this must be true-- the calls are a spanning set. There's another way to look at it, which is-- I just-- like from calculus, where we can actually make explicit what this spanning-- what this portfolio of calls looks like in the arbitrary case.

So let me just do that. So you can sort of see, there must be a linear combination by this for a piecewise linear. Therefore in the limit, any continuous function must be able to be replicated by calls. How many of each?

Well, there's actually a very, very simple result. That is as follows-- and, well, let's just write down an exact Taylor series to the second order. So this is-- so for any function with second derivative, let's just write down a Taylor series-- the first two terms.

And let's put the second term-- we can just do an exact second-order term, so 0 to infinity makes minus c plus g double prime of c dc . c is my dummy variable. Actually, I've gone to plus notation. Here's the max of this and 0.

OK, that's an exact Taylor series, true for any-- it's not an approximation. That's exact. You just integrate the right-hand side by parts if you want to verify it.

Maybe it's obvious to you, but I'm so used to just doing not exact Taylor series. So this is the second order. So this holds for any g exactly.

And now I'm just going to make one little change, which sort of might make obvious what we're trying to do. I'm just going to take this dummy variable c , which we're integrating over from 0 to infinity, and just call it K . We can certainly do that.

All right, this now looks like the payout of a call. It's the payout of the call price. Now, I don't want to be integrating.

Remember, if I want to actually get the call price, I take the expected value of this. I integrate x over x with respect to its density. This is g of a payout function of x . Here I'm integrating over K , so I'm doing something a bit different.

But this is the call option payout. So this holds. It's a linear equation, obviously. And of course, expectation is a linear operator.

So I'm just going to take, well, what are the two steps? First of all, I'm just going to replace x with my random variable $S_{sub} T$. So that I can do. This also holds.

And formally, of course, $S_{sub} T$ is a random variable, so it's a function from the sample space of the real line. But this holds for every point on the sample space. So I can write down this equation between random variables.

Here it's just the integral over dK . So that hold. Now I'm going to take the expectation operator. So take discounted expected value, of each side.

So in other words, what is my operator [INAUDIBLE]? It looks like Z_{tT} , expected value of given ST . All right? OK, so this one is a discount expected value. That's the price.

So this becomes price of the derivative with payout at maturity g . All right, what do we have here? Well, first we've got a constant. So we've got a constant times-- OK, so that's a constant.

OK, now we've got the discounted expected stock price. A little bit of thought on the terms of the Fundamental Theorem will show you that the discounted expected stock price under this operator is the current stock price. It's actually non-trivial, but just think of the stock itself as a derivative, with the payout S , and apply the Fundamental Theorem.

This has to be the case, because a replicating portfolio of the stock is just a holding of the stock. Plus-- and then we just take the integral. So the expectation inside the integral-- OK, so now I've got discounted expected payout of this. And the discounted expected payout of this is just the call price, with strike K .

OK, so I really like this formula. In some sense, there's nothing too complicated about how to derive it. But it says explicitly now, how do I replicate an arbitrary derivative product with payout g of x or g of S at maturity? Well, it's clear. I replicate it by g_0 , zero coupon bonds.

So I have g_0 of zero coupon bonds. That's this. I have g' zero stock-- that's this. And I have this linear combination of calls.

So there-- this kind of makes sense, right? You want the zero coupon bond amount is just the intercept of g . The number of stocks is just the slope of g at 0. And then I have this linear combination of call prices.

I've just proved that by taking this, and taking expected values. So this is sort of looking at the duality of option prices and probabilities in different ways. But then, also how calls span everything. So the calls, in some sense, are the primitive information.

Once I know all call option prices, I know the probability distribution exactly. So there are a couple of sort of interesting further questions you might want to pose. We seem to have done everything here with regard to the distribution at time capital T .

And that's true. I know all the calls. I know the distribution at time capital T . I know all the calls. I know the price of any option with a payout defined solely by a function at capital T .

But I said nothing about the path that takes the stock from today until capital T. So I'm just going to leave you with two things to think about. Actually, it's one thing to think about. Two people thought about a lot.

And it's the following question, which now we'll start transitioning into stochastic calculus, and stochastic processes a little bit. So we know-- let's just imagine two times. So suppose we know-- so we know the set of all call prices with maturity T1, for all K, and the set of all call prices with maturity T2 for all K.

OK, so then we know the distribution. Well, there are two distributions. We know the distribution of T1 given ST, and-- but do we know the distribution of the stock at T2 given T1? More of a general point-- suppose I know this for all T.

Let's put T0 here. OK, I know all option prices of all maturities and all strikes. Can I determine the stochastic process for ST over this time?

Is the underlying stochastic process for the stock price fully determined by knowing all call option prices for all strikes and all maturities? The marginal distributions or the conditional distributions for all maturities are determined, because we know that here. Well, you'll probably see this is a rephrasing of a finite dimensional problem from probability. The answer is no.

And the reason to think about is, if I know all the-- my intuition for this is if I know all the distributions that-- think about just a denser and denser grid of times that I know the distribution of-- getting closer and closer. I can still allow the stock to flip instantaneously quickly. Imagine they're all essentially symmetric distributions, and they're all roughly the same expanding out.

I can imagine the stock flipping discontinuously over an arbitrarily small time interval. So without a constraint on the continuity of this process, or mathematical constraints on this process, you can't determine the actual process for the stock, even given all the option prices-- call option prices. So there are two-- so Emanuel Derman, who was at Goldman Sachs, now at Columbia-- and Bruno Dupire-- who's, I think, still at Bloomberg-- this is the early '90s-- basically determined the conditions

that you need.

And the basic conditions are that just the stock has to be a diffusion process. If it is a diffusion process-- the random term is Brownian motion-- then it is, actually, fully determined. And it's a really nice, elegant result.

So this is what gets mathematically quite nice, and a little tricky. But there's a practical implication of this, as well, which is in practice, I will know a finite subset of call options. Those prices will be available to me in the market. So they will be given.

So one thing I know for sure is that even with a very densely set of call option prices, there will be some other derivative prices whose price is not exactly determined by that set of calls. Because in particular, I know that the set of calls does not determine the underlying stochastic process, even if I knew all of them. So that's a very important thing for traders to understand, is that even if I know a lot of market information-- so I'm given here are the prices of a large number of European options, European call options I can trade-- there may be a complex or nonstandard derivative product, whose price is not determined uniquely, simply by knowing those options.

And that is one of the challenges for some of the quant groups. So anyway, with that, that is all I wanted to convey. I'm happy to take some questions. And thank you for your time. Thank you for having me. I appreciate it.

[APPLAUSE]

AUDIENCE: Yeah, I have-- I was just wondering, so set the call, or the set of all calls basically spans the space of all possible payoffs, right?

STEPHEN Yes.

BLYTHE:

AUDIENCE: So I was just wondering if maybe if we could change, and select some other such basis for spanning it? Instead of call, maybe some other kind of basic payoff that could still span the same thing, and maybe it's more easily credible, or something?

STEPHEN BLYTHE: Yeah, that's a good-- there must be many, if I can-- but this, given that this is the simplest expansion of the function g , an arbitrary function g , and the second term comes in with this call payout, gives us this elegance. Of course, if I know all the digitals, I know the cumulative distribution function, and therefore, I know the density. So I mean, the digitals do the same. And in fact, Arrow-Debreu securities, which is our building blocks, which is something that pays off one in a particular state, sample state, also are building blocks.

AUDIENCE: [INAUDIBLE].

STEPHEN BLYTHE: I mean, sometimes, you could think about an arbitrary basis that will span-- an arbitrary basis of functions that will span any continuous function. And sometimes, you can do it in any polynomial expansion. If I have a price and any of those payouts, and I've got my spanning set. But this is the most elegant one. Yeah, next question there.

AUDIENCE: I have a question about the last [INAUDIBLE] mentioned. [INAUDIBLE] because market's incomplete, so you can not sort of use call option to replicate the stock itself.

STEPHEN BLYTHE: You can use a call option to replicate a stock. As long as you have zero coupon. You can see from here, I can just reorganize everything here to zero coupon bond stock, and a set of calls will span anything-- with maturity T . What they're sort of saying is, if I have this strange process with jumps, and flips, and discontinuities, then the market is incomplete, I guess is what this is saying.

AUDIENCE: OK, yeah, so [INAUDIBLE] is due to the inconvenience.

STEPHEN BLYTHE: Yeah, in the sense of most finance-- in fact, all continuous time finance will assume there's some diffusion process for some process for stock, which has some Brownian motion. There's some function here, and some function for the drift term. In that case, then all the call prices do determine. If you think there's some exogenous flipping parameter, that's my intuition for it. So there's some-- that's why this is incomplete. So this will not determine.

So in particular, I could know all these call prices. Then I could determine a particular derivative product. It could be the number of times that in an arbitrarily small interval, the stock flips this many times. I mean, there's some-- you can create whatever you like for a derivative that would be incomplete for these calls.

AUDIENCE: So in this case, go back to a previous question as we just mentioned-- the second-order derivative of a call option with respect to a strike is [INAUDIBLE] neutral density. So in this case, it was not-- that reason [INAUDIBLE] particular instance of that [INAUDIBLE] is not uniquely determined.

STEPHEN BLYTHE: No, the risk neutral density is uniquely determined. The stochastic process for S_T over all time is not uniquely determined. So this is uniquely determined by call option prices. That is uniquely determined.

But knowing the conditional distribution of S_T , given S_t doesn't determine the process of the stock price. To get there-- I can think of infinitely many processes of the stock price that can give rise to this distribution. That's what's not determined. The terminal distribution is uniquely determined by the call option prices-- nothing else.

AUDIENCE: So in this case, if we take Z over θ , so we'll get a particular risk neutral density for each particular stock?

STEPHEN BLYTHE: That's correct. Right, thank you very much. Appreciate it.

BLYTHE: