

Lecture 21: Stochastic Differential Equations

In this lecture, we study stochastic differential equations. See Chapter 9 of [3] for a thorough treatment of the materials in this section.

1. STOCHASTIC DIFFERENTIAL EQUATIONS

We would like to solve differential equations of the form

$$dX = \mu(t, X(t))dt + \sigma(t, X(t))dB(t)$$

for given functions a and b , and a Brownian motion $B(t)$. A function (or a path) X is a solution to the differential equation above if it satisfies

$$X(T) = \int_0^T \mu(t, X(t))dt + \int_0^T \sigma(t, X(t))dB(t).$$

Following is a quote from [3].

Stochastic differential equations provide a link between probability theory and the much older and more developed fields of ordinary and partial differential equations. Wonderful consequences flow in both directions. The stochastic modeler benefits from centuries of development of the physical sciences, and many classic results of mathematical physics (and even pure mathematics) can be given new intuitive interpretations.

We first state a result saying that SDEs can be solved.

Theorem 1.1. (*Existence and uniqueness*) *If the coefficients of the stochastic differential equation*

$$dX = \mu(t, X(t))dt + \sigma(t, X(t))dB(t), \quad \text{with } X(0) = x_0 \quad \text{and} \quad 0 \leq t \leq T,$$

satisfy a space-variable Lipschitz condition

$$|\mu(t, x) - \mu(t, y)|^2 + |\sigma(t, x) - \sigma(t, y)|^2 \leq K|x - y|^2$$

and the spatial growth condition

$$|\mu(t, x)|^2 + |\sigma(t, x)|^2 \leq K(1 + |x|^2),$$

then there is a continuous adapted solution $X(t)$ such that (L^2 bound). Moreover, if $X(t)$ and $Y(t)$ are both continuous solutions satisfying the L^2 bound, then

$$\mathbf{P}(X(t) = Y(t) \text{ for all } t \in [0, T]) = 1.$$

The proof of this theorem is quite technical and can be found in [3]. Thanks to this theorem, we know that most SDEs in fact have a solution. We now discuss some simple (but important) examples of SDEs which have closed form solutions.

1.1. Coefficient matching method. One of the most natural, and most important, stochastic differential equations is given by

$$dX(t) = \mu X(t) dt + \sigma X(t) dB(t) \quad \text{with } X(0) = x_0 > 0,$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are constants.

Let us pretend that we do not know the solution and suppose that we seek a solution of the form $X(t) = f(t, B(t))$. For this candidate, we have

$$dX(t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \frac{\partial f}{\partial x} dB(t),$$

hence if we must have

$$\mu f = \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \quad \text{and} \quad \sigma f = \frac{\partial f}{\partial x}.$$

The second equation gives $f(t, x) = e^{\sigma x + g(t)}$. Using this in the first equation gives

$$\mu f = g'(t)f + \frac{\sigma^2}{2} f.$$

Therefore, $g'(t) = \mu - \frac{\sigma^2}{2}$, and we see that

$$f(t, x) = x_0 e^{\sigma x + (\mu - \sigma^2/2)t}.$$

Therefore, $X(t) = x_0 e^{(\mu - \sigma^2/2)t + \sigma B(t)}$.

1.2. Coefficient matching for product processes. Let α and σ be positive constants and consider the following SDE

$$dX(t) = -\alpha X(t) dt + \sigma dB(t) \quad \text{with } X(0) = x_0.$$

Ornstein and Uhlenbeck first used (a version of) this equation to study the behavior of gasses. It has been applied (or rediscovered) in a variety of contexts. This SDE exhibits the ‘mean reversion’ behavior (when $\alpha > 0$).

Coefficient matching method fails for this SDE, so we try a different test function

$$X(t) = a(t) \left(x_0 + \int_0^t b(s) dB(s) \right),$$

where $a(0) = 1$. By differentiating each side we get,

$$dX(t) = \frac{a'(t)}{a(t)} X(t) dt + a(t)b(t) dB(t),$$

where we assume that $a(t) > 0$ for all t . This should match the given SDE, so we must have

$$-\alpha = \frac{a'(t)}{a(t)} \quad \text{and} \quad \sigma = a(t)b(t).$$

Therefore, $a(t) = e^{-\alpha t}$ and $b(t) = \sigma e^{\alpha t}$. From this, we see that

$$X(t) = x_0 e^{-\alpha t} + \int_0^t \sigma e^{\alpha(s-t)} dB(s).$$

2. NUMERICAL METHODS

Most PDE and SDE do not have closed form solutions. In this case we can use numerical methods such as finite difference method, tree method, or Monte Carlo simulation to find an approximate solution. We will briefly discuss the some of the methods.

2.1. Finite difference methods. Here is an example of using finite difference method in solving an ordinary differential equation.

Example 2.1. Suppose we want to solve $u'(x) = 5u(x) + 2$, $u(0) = 0$ to compute $u(1)$.

Step 1. Choose a small value of h . Our plan is to compute the (approximate) value of u at the points $x = 0, h, 2h, 3h, 4h, 5h, \dots, kh$ where $kh = 1$. We hope that the numerical value approaches the real value as we take smaller values of h . (For example, take $h = 1/2$)

Step 2. Use Taylor's formula to compute

$$\begin{aligned} u((i+1)h) &\approx u(ih) + h \cdot u'(ih) \\ (2.1) \qquad \qquad &= u(ih) + h \cdot (5u(ih) + 2). \end{aligned}$$

Note that $u((i+1)h)$ can be computed approximately based on the value of $u(ih)$. Hence we can continue computing the values of u at our sample points.

For $h = 1/2$, we see that $u(\frac{1}{2}) \approx u(0) + \frac{1}{2}u'(0) = u(0) + \frac{1}{2}(5 \cdot u(0) + 2) = 1$, and $u(1) \approx u(\frac{1}{2}) + \frac{1}{2}u'(\frac{1}{2}) = u(\frac{1}{2}) + \frac{1}{2}(5u(\frac{1}{2}) + 2) = \frac{9}{2}$.

This method can easily be extended to partial differential equations. For example, when studying a function $u(x, y)$ of two variables, we may compute the value of u at the intersection points of some fine grid, i.e., we choose some small real h , and compute the values $u(ih, jh)$ for integers $i = 0, 1, 2, \dots$ and $j = 0, 1, 2, \dots$.

This method cannot be directly applied to solve SDEs. This is because in SDEs, the equation corresponding to (2.1) involves random variables.

2.2. Monte Carlo simulation. Monte Carlo simulation is a method that is used to simulate a probability space by taking independent samples from the space according to the probability distribution.

Monte Carlo simulation can be used to resolve this issue for SDEs. Suppose that we have a SDE of the form

$$df(t, B_t) = g(t, B_t)dB_t + h(t, B_t)dt.$$

If we already know the path B_t , then the equation now becomes a PDE, hence one can use the finite difference method to solve it.

Monte Carlo simulation (applied to this setting) involves the following three steps.

Step 1: Choose a random sample path B_t according to the probability distribution.

Step 2: Use the sample path from Step 1 and finite difference method to solve the SDE for the particular choice of sample path.

Step 3: Repeat Step 1 and 2 many times.

This gives a probability distribution of the random stochastic process $f(t, B_t)$. Monte Carlo simulation is based on the idea that the resulting probability distribution of this method will converge to the distribution of the stochastic process that solves the SDE.

2.3. Tree method. Tree method uses the idea that the Brownian motion can be seen as a limit of a simple random walk. Suppose that we would like to compute the value of $f(t, B_t)$ at some time $t = T$, where

$$df(t, B_t) = g(t, B_t)dB_t + h(t, B_t)dt.$$

We begin by taking sample points t_0, t_1, t_2, \dots of the time domain $[0, T]$.

We replace the occurrences of B_t in the SDE by a simple random walk which either goes up one step or down one step during each time interval $[t_i, t_{i+1}]$ (where the step size is appropriately chosen). This gives an inductive way to approximately find the probability distribution of $f(T, B_T)$.

Hull [4] illustrates how these methods are used in financial applications.

3. HEAT EQUATION

Our last topic of study is a well-known PDE, heat equation. It is well known that the Black-Scholes equation can be turned into a heat equation after a suitable change of variables.

Let $u(x, t)$ be a function of two variables, space and time (denoted x and t). The following differential equation is known as the one dimensional heat equation (diffusion equation):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

This is one of the few partial differential equations that is very well understood (and has a closed form solution).

Example 3.1. Let $u(x, t)$ represent the temperature in a long, thin, uniform bar of material whose sides are perfectly insulated so that its temperature varies only with distance x along the bar (and with time t). Then $u(x, t)$ satisfies the *heat equation* (this is where the name of the equation comes from).

Our goal is to solve various initial value problems for the heat equation. The initial values that we consider will be given as

$$u(0, x) = u_0(x) \quad (\text{for } -\infty < x < \infty),$$

for some function u_0 .

Observation 1. Heat equation is linear, i.e., if $u_1(x, t)$ and $u_2(x, t)$ satisfies the heat equation, then $(u_1 + u_2)(x, t)$ also satisfies the heat equation. More generally if we have a collection of solutions $u_s(x, t)$ indexed by $s \in \mathbb{R}$, then $\int_{-\infty}^{\infty} u_s(x, t) \cdot c(s) ds$ is also a solution (as long as the integral exists and is differentiable up to appropriate order). This means that we can superimpose solutions of ‘easy’ initial value problems to obtain a solution to a more general initial value problem.

Observation 2. The ‘easy’ initial value problem we are going to use is when the initial value is given as a Dirac delta function. Let $\delta(x)$ be the Dirac delta function and suppose that $u_0 = \delta$ so that we are solving

$$u(0, x) = \delta(x).$$

The solution for this initial value problem is known to be

$$u_\delta(x, t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/(4t)} \quad (\text{for } -\infty < x < \infty, t > 0).$$

Note that the solution ‘converges to’ the Dirac delta function as t tends to zero. Also note that for fixed value of $t > 0$, this is a probability distribution function of the normal random variable.

Exercise 3.2. Derive the solution above by using $\xi = \frac{x}{\sqrt{t}}$, and $U(\xi) = t^{1/2}u(x, t)$, and restating the heat equation as an ODE.

Now suppose that a function u_0 is given. We can understand u_0 as a function obtained by superimposing Dirac delta functions, i.e.,

$$u_0(x) = \int_{-\infty}^{\infty} \delta(x - s) u_0(s) ds.$$

Consider the following function obtained by superimposing the solutions accordingly:

$$u(x, t) = \int_{-\infty}^{\infty} u_\delta(x - s, t) \cdot u_0(s) ds$$

(such function need not be well-defined since the integration might not exist).

However, if the function u_0 is ‘reasonable’ then we can show that

$$(3.1) \quad \frac{\partial u}{\partial t}(x, t) = \int_{-\infty}^{\infty} \frac{\partial u_\delta}{\partial t}(x - s, t) \cdot u_0(s) ds \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}(x, t) = \int_{-\infty}^{\infty} \frac{\partial^2 u_\delta}{\partial x^2}(x - s, t) \cdot u_0(s) ds.$$

This implies that $u(x, t)$ satisfies the heat equation as well. Note that $u_0(x, 0) = u_0(x)$, and hence as long as a ‘reasonable’ initial condition u_0 is given (so that $u(x, t)$ is well-defined and (3.1) holds), we see that $u(x, t)$ solves the initial value problem.

Lecture 21

6

REFERENCES

- [1] P. Wilmott, S. Howison, J. Dewynne, The mathematics of financial derivatives
- [2] S. Shreve, Stochastic calculus for finance II: continuous-time models
- [3] M. Steele, Stochastic calculus and financial applications
- [4] J. Hull, Options, futures, and other derivatives

MIT OpenCourseWare
<http://ocw.mit.edu>

18.S096 Topics in Mathematics with Applications in Finance
Fall 2013

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.