Risk Neutral Pricing
Black-Scholes Formula
Lecture 19

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Risk Neutral Valuation: Two-Horse Race Example

- One horse has 20% chance to win another has 80% chance

- $10000 is put on the first one and $50000 on the second

If odds are set 4-1:

- Bookie may gain $10000 (if first horse wins)
- Bookie may loose $2500 (if second horse wins)
- Bookie expects to make 0.2 * (10000) + 0.8 * (-2500) = 0

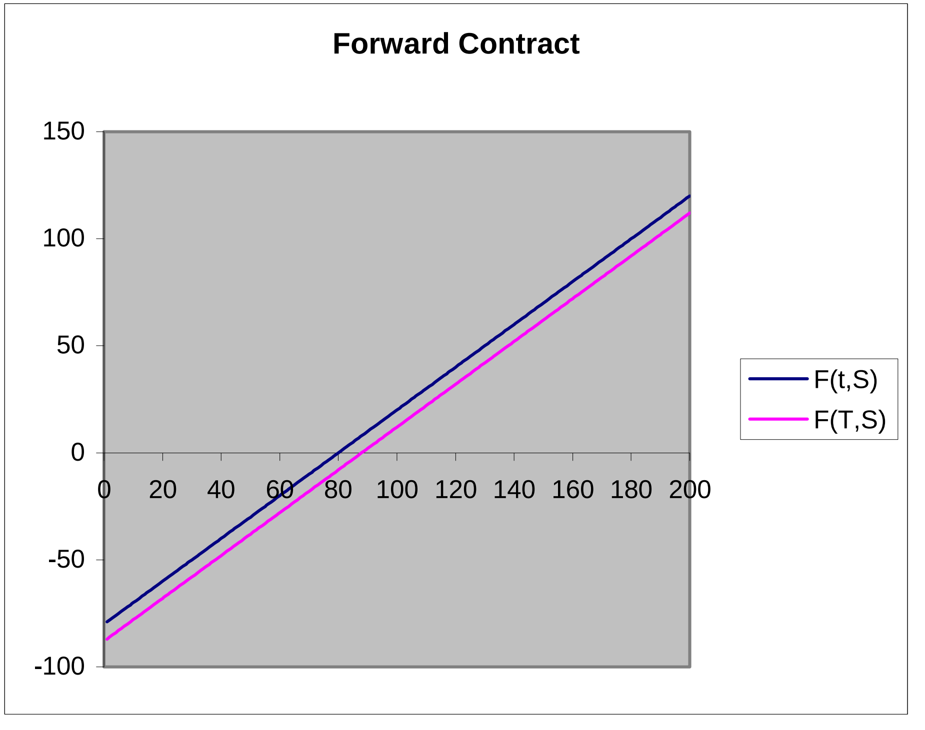
If odds are set 5-1:

- Bookie will not lose or gain money no matter which horse wins

Risk Neutral Valuation : Introduction

We are interested in finding prices of various *derivatives*.

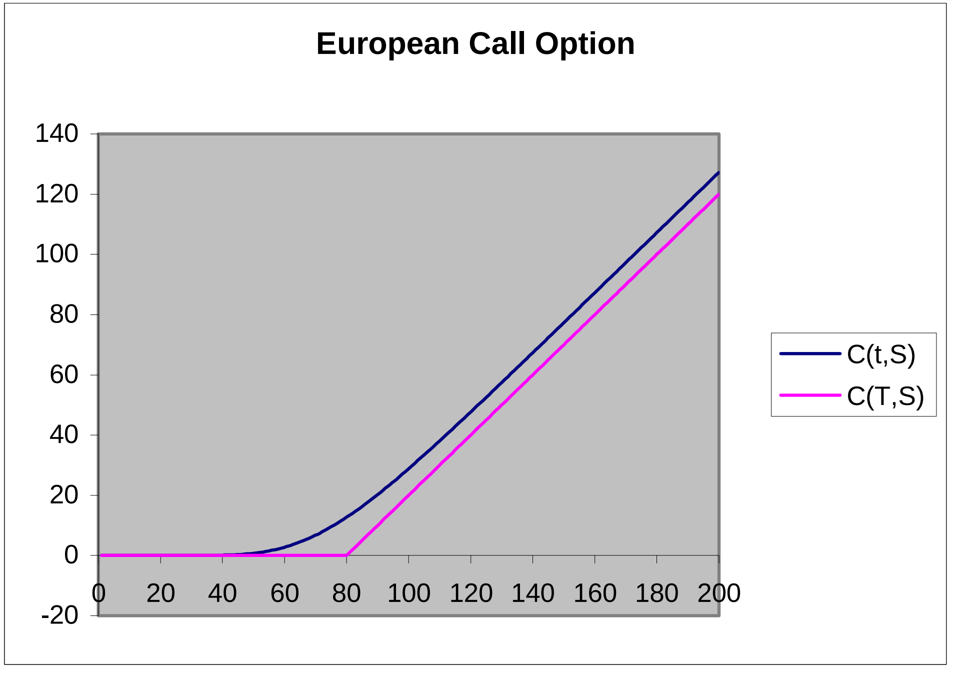
Forward contract pays S-K at time T :



$S(t)=80, K=88.41, T=2$ (years)

Risk Neutral Valuation: Introduction

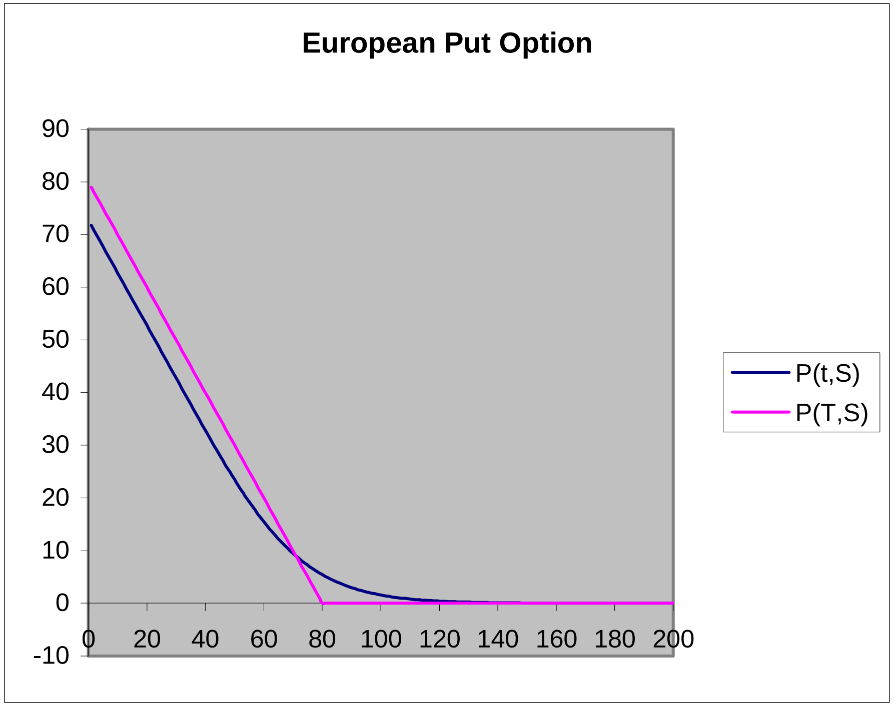
European Call option pays $max(S\text{-}K,0)$ at time T



$S(t)=80$, $K=80$, $T=2$ (years)

Risk Neutral Valuation: Introduction

European Put option pays $max(K\text{-}S,0)$ at time T



$S(t)=80,\ K=80,\ T=2$ (years)

Risk Neutral Valuation: Introduction

- Given current price of the stock and assumptions on the dynamics of stock price, there is no uncertainty about the price of a derivative
- The price is defined only by the price of the stock and not by the risk preferences of the market participants
- Mathematical apparatus allows to compute current price of a derivative and its risks, given certain assumptions about the market

Risk Neutral Valuation: Replicating Portfolio

Consider *Forward* contract which pays S-K in time dt. One could think that its strike K should be defined by the "real world" transition probability p:

$$p(S_1-K)+(1-p)(S_2-K)=pS_1+(1-p)S_2-K$$

$$K_0= pS_1+(1-p)S_2$$

If $p=1/2$, $K_0=(S_1+S_2)/2$

Risk Neutral Valuation: Replicating Portfolio

Consider the following strategy:

1. Borrow $\$S_0$ to buy the stock. Enter *Forward* contract with strike K_0

2. In time dt deliver stock in exchange for K_0 and repay $\$S_0 e^{rdt}$

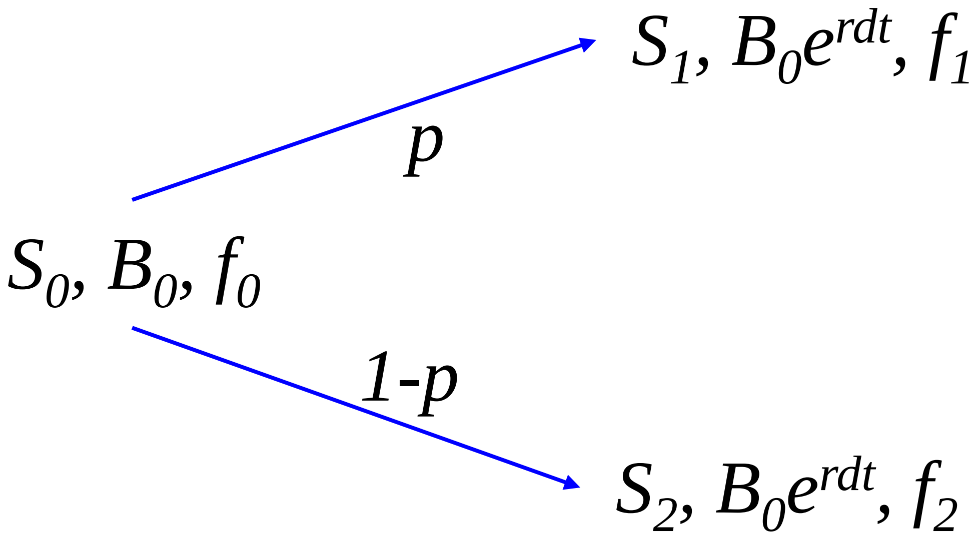
- If $K_0 > S_0 e^{rdt}$ we made riskless profit
- If $K_0 < S_0 e^{rdt}$ we definitely lost money

$$\Rightarrow K_0 = S_0 e^{rdt}$$

Current price of a derivative claim is determined by current price of a portfolio which exactly replicates the payoff of the derivative at the maturity

Risk Neutral Valuation: One step binomial tree

Suppose our economy includes stock S, riskless money market account B with interest rate r and derivative claim f. Assume that only two outcomes are possible in time dt:



Risk Neutral Valuation: One step binomial tree

For a general derivative claim f, find a and b such that

$$f_1 = aS_1 + bB_0 e^{rdt}$$
$$f_2 = aS_2 + bB_0 e^{rdt}$$

Then

$$f_0 = aS_0 + bB_0$$

Easy to see that

$$a = \frac{f_1 - f_2}{S_1 - S_2}, b = \frac{S_1 f_2 - S_2 f_1}{(S_1 - S_2) B_0 e^{rdt}}$$

$$f_0 = e^{-rdt}\left(S_0 e^{rdt} \frac{f_1 - f_2}{S_1 - S_2} + \frac{S_1 f_2 - S_2 f_1}{S_1 - S_2} \right)$$

Risk Neutral Valuation: One step binomial tree

One should notice that

$$f_0 = e^{-rdt}\left(f_1 \frac{S_0 e^{rdt} - S_2}{S_1 - S_2} + f_2 \frac{S_1 - S_0 e^{rdt}}{S_1 - S_2} \right)$$

$$f_0 = e^{-rdt}(f_1 q + f_2(1 - q))$$

where

$$q=(S_0 e^{rdt}-S_2)/(S_1-S_2), \ \ 0<q<1$$

Moreover

$$S_1 q+S_2(1-q)= e^{rdt} S_0$$

Risk Neutral Valuation: Continuous case

$$f_t = e^{-r(T-t)} E_Q[f_T]$$

Q is the risk neutral (martingale) measure under which

$$S_0 = e^{-rt} E_Q[S_t]$$

Black-Scholes equation

Assume that the stock has log-normal dynamics:

$$dS = \mu S dt + \sigma S dW$$

Where dW is normally distributed with mean 0 and standard deviation $\sqrt{dt}$ (i.e. W is a Brownian Motion)

We want to find a replicating portfolio such that

$$df = a dS + b dB$$

Black-Scholes equation

Use *Ito's formula:*

$$df(S,t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS)^2$$

$$(dS)^2 = \sigma^2 S^2 dt$$

(analogous to first order Taylor expansion, up to dt term)

Black-Scholes equation

$$df = a\,dS + b\,dB$$

Substitute dS, df, $dB = rB\,dt$ and $(dS)^2$

$$\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S\,dW = (a\mu S + brB)dt + a\sigma S\,dW$$

Compare terms

$$a = \frac{\partial f}{\partial S},\ brB = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2$$

Black-Scholes equation

$bB=f\text{-}aS$ is deterministic and as $dB=rBdt$

$$d(f\text{-}aS)=r(f\text{-}aS)dt$$

Substituting once again $df=\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial S}dS+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 dt$ and $a=\frac{\partial f}{\partial S}$

we obtain the **Black-Scholes equation**

$$\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2+\frac{\partial f}{\partial S}rS-rf=0$$

Fisher Black, Myron Scholes – paper 1973

Myron Scholes, Robert Merton – Nobel Prize 1997

Black-Scholes equation

- Any tradable derivative satisfies the equation
- There is no dependence on actual drift μ
- We have a hedging strategy (replicating portfolio)
- By a change of variables Black-Scholes equation transforms into heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

Black-Scholes equation

Boundary and *final* conditions are determined by the pay-off of a specific derivative

For European Call

$$C(S,T)=max(S\text{-}K,0)$$

$$C(0,t)=0, C(\infty,t)\cong S$$

For European Put

$$P(S,T)=max(K\text{-}S,0)$$

$$P(0,t)=Ke^{-r(T-t)}, P(\infty,t)=0$$

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Black-Scholes equation

For European Call/Put the equation can be solved analytically

$$C_t = e^{-r(T-t)}\left(e^{r(T-t)}SN(d_1) - KN(d_2)\right)$$

$$P_t = e^{-r(T-t)}\left(KN(-d_2) - e^{r(T-t)}SN(-d_1)\right)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du$$

Black-Scholes: Risk Neutral Valuation

$$f_t = e^{-r(T-t)} E_Q[f_T]$$

Q is the risk neutral measure under which

$$dS = rSdt + \sigma SdW$$

$$PDF(S_T) = \frac{1}{\sigma S\sqrt{2\pi T}} \exp\left(-\frac{(\ln(S_T / S_t) - (r - \sigma^2/2)(T-t))^2}{2\sigma^2 (T-t)} \right)$$

Black-Scholes equation

For more complicated options or more general assumptions numerical methods have to be used:

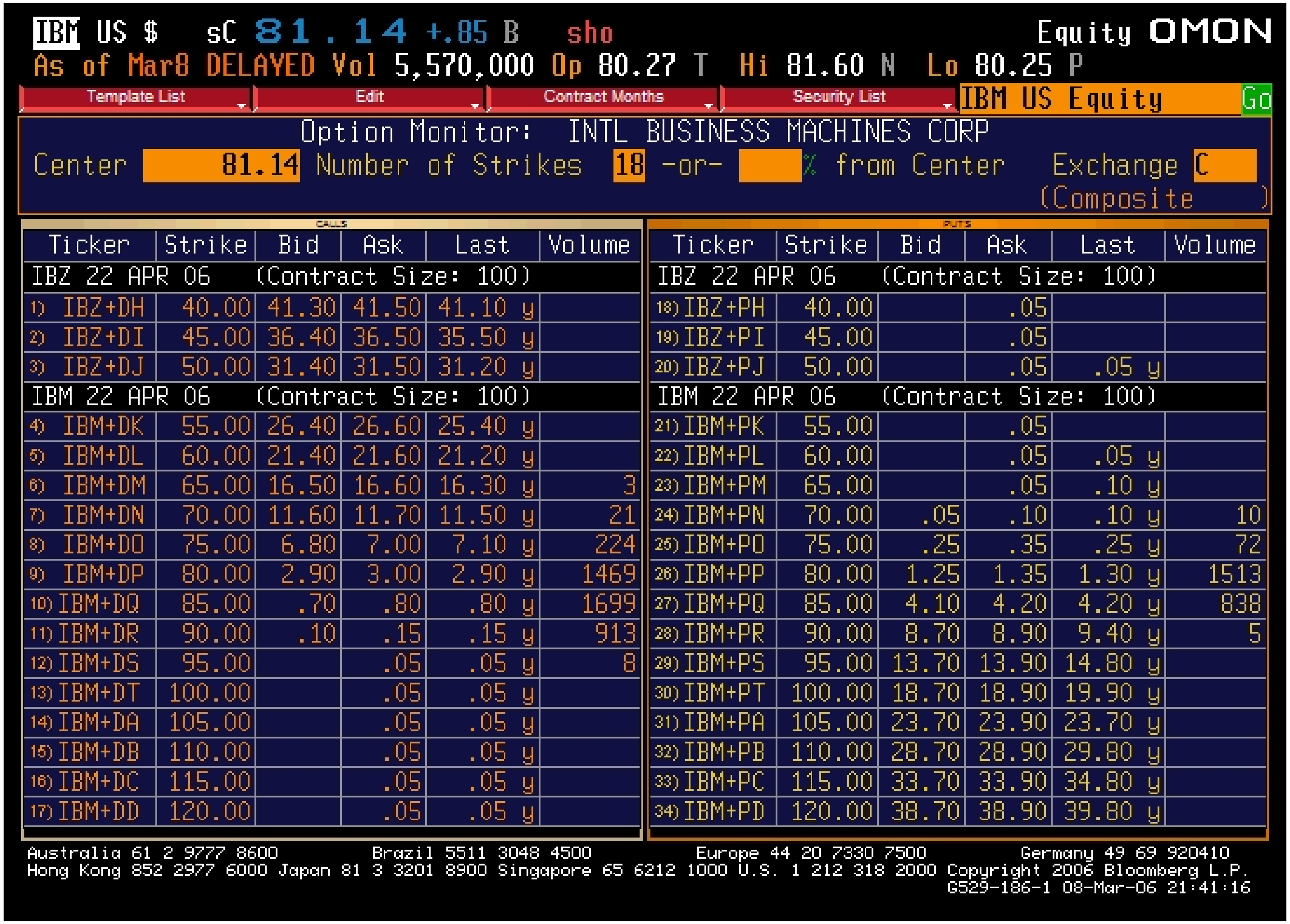
- Finite difference methods
- Tree methods (equivalent to explicit scheme)
- Monte Carlo simulations

Black-Scholes equation: Conclusions

Modern financial services business makes use of

- PDE
- Numerical methods
- Stochastic Calculus
- Simulations
- Statistics
- Much, much more

Risk Neutral Valuation: Example



IBM US $ sC 81.14 +.85 B sho Equity OMON

As of Mar8 DELAYED Vol 5,570,000 Op 80.27 T Hi 81.60 N Lo 80.25 P

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Option Monitor: INTL BUSINESS MACHINES CORP

Center 81.14 Number of Strikes 18 -or- % from Center Exchange C (Composite)

CALLS

	Ticker	Strike	Bid	Ask	Last		Volume
	IBZ 22 APR 06 (Contract Size: 100)						
1)	IBZ+DH	40.00	41.30	41.50	41.10	y	
2)	IBZ+DI	45.00	36.40	36.50	35.50	y	
3)	IBZ+DJ	50.00	31.40	31.50	31.20	y	
	IBM 22 APR 06 (Contract Size: 100)						
4)	IBM+DK	55.00	26.40	26.60	25.40	y	
5)	IBM+DL	60.00	21.40	21.60	21.20	y	
6)	IBM+DM	65.00	16.50	16.60	16.30	y	3
7)	IBM+DN	70.00	11.60	11.70	11.50	y	21
8)	IBM+DO	75.00	6.80	7.00	7.10	y	224
9)	IBM+DP	80.00	2.90	3.00	2.90	y	1469
10)	IBM+DQ	85.00	.70	.80	.80	y	1699
11)	IBM+DR	90.00	.10	.15	.15	y	913
12)	IBM+DS	95.00		.05	.05	y	8
13)	IBM+DT	100.00		.05	.05	y	
14)	IBM+DA	105.00		.05	.05	y	
15)	IBM+DB	110.00		.05	.05	y	
16)	IBM+DC	115.00		.05	.05	y	
17)	IBM+DD	120.00		.05	.05	y	

PUTS

	Ticker	Strike	Bid	Ask	Last		Volume
	IBZ 22 APR 06 (Contract Size: 100)						
18)	IBZ+PH	40.00		.05			
19)	IBZ+PI	45.00		.05			
20)	IBZ+PJ	50.00		.05	.05	y	
	IBM 22 APR 06 (Contract Size: 100)						
21)	IBM+PK	55.00		.05			
22)	IBM+PL	60.00		.05	.05	y	
23)	IBM+PM	65.00		.05	.10	y	
24)	IBM+PN	70.00	.05	.10	.10	y	10
25)	IBM+PO	75.00	.25	.35	.25	y	72
26)	IBM+PP	80.00	1.25	1.35	1.30	y	1513
27)	IBM+PQ	85.00	4.10	4.20	4.20	y	838
28)	IBM+PR	90.00	8.70	8.90	9.40	y	5
29)	IBM+PS	95.00	13.70	13.90	14.80	y	
30)	IBM+PT	100.00	18.70	18.90	19.90	y	
31)	IBM+PA	105.00	23.70	23.90	23.70	y	
32)	IBM+PB	110.00	28.70	28.90	29.80	y	
33)	IBM+PC	115.00	33.70	33.90	34.80	y	
34)	IBM+PD	120.00	38.70	38.90	39.80	y	

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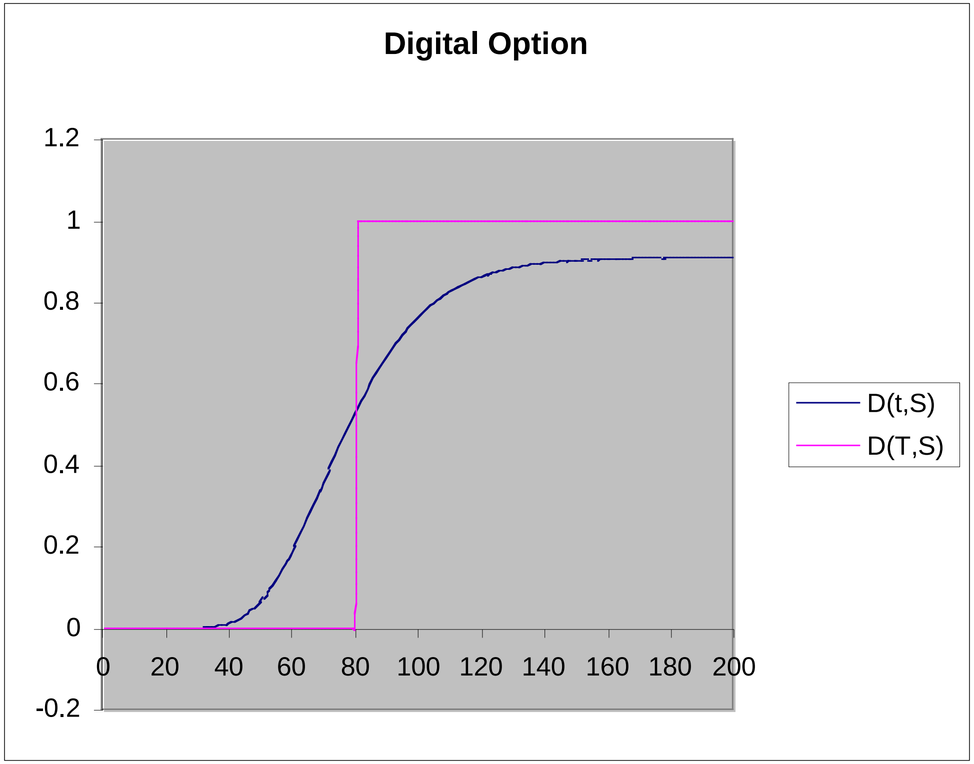
Risk Neutral Valuation: Example

Digital option pays *1* if $S>K$ at time T



$S(t)=80$, $K=80$, $T=2$ (years)

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