18.S096 Problem Set 6 Fall 2013
Time Series II and Portfolio Theory
Due Date: 11/7/2013

1. Suppose $\boldsymbol{X}_t = \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix}$ follows a $VAR(1)$ model where

   $$X_{1,t} = 0.3 + .8 \cdot X_{1,t-1} + \epsilon_{1,t}$$
   $$X_{2,t} = 0.2 + .6 \cdot X_{1,t-1} + .4 \cdot X_{2,t-1} + \epsilon_{2,t}$$

   where $(\epsilon_{1,t}, \epsilon_{2,t})^T$ are $i.i.d.$ $N(0_2, \Sigma)$, and

   $$\Sigma = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

   1(a) Compute $\mu = E[\boldsymbol{X}_t]$.

   1(b) Compute $\Gamma(0) = Cov[\boldsymbol{X}_t]$.

   1(c) Compute $\Gamma(1) = Cov[\boldsymbol{X}_t, \boldsymbol{X}_{t-1}]$

   1(d) Derive a formula for computing $\Gamma(h) = Cov[\boldsymbol{X}_t, \boldsymbol{X}_{t-h}]$, $h \geq 1$.

2. For $\{\epsilon_t\}$ i.i.d. $WN(0, \sigma^2)$, define processes $\{w_t\}$ and $\{v_t\}$ as follows

   $$w_t = 5(1 - .5L)^{-1}\epsilon_t$$
   $$v_t = 4(1 - .4L)^{-1}\epsilon_t$$

   Define $\{x_t\} : x_t = w_t - v_t$.

   2(a) Solve for coefficients θ_i in the infinite-order moving average process for x_t :

   $$x_t = \epsilon_t + \sum_{i=1}^{\infty} \theta_t \epsilon_t$$

   2(b) Prove that $\{x_t\}$ is an $AR(2)$ process.

   2(c) Solve for ϕ_1 and ϕ_2 in the representation:

   $$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \epsilon_t$$

   2(d) Prove that any stationary $AR(2)$ process can be expressed as the difference of two (possibly infinite order) moving average processes on the same innovation process $\{\epsilon_t\}$.

3. Consider a single-period analysis of 2 risky assets with

   Returns:

   $$\boldsymbol{R} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

   Mean and Covariance of Returns:

   $$E[R] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \text{ and } Cov[\boldsymbol{R}] = \Sigma = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{1,2} & \Sigma_{2,2} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

   where $\sigma_1 = \sqrt{\Sigma_{1,1}}$, $\sigma_2 = \sqrt{\Sigma_{2,2}}$ and ρ is the correlation between R_1 and R_2.

   A portfolio $\boldsymbol{w} = (w_1, w_2)^T$ identified by its investment weights in the assets, has return:

   $$R_w = w_1 R_1 + w_2 R_2 = \boldsymbol{w}^T \boldsymbol{R}.$$

   Assume that w is fully invested ($w_1 + w_2$=1) and that no short sales are allowed ($w_1 \geq 0$ and $w_2 \geq 0$).

   (3a) Prove that $Var(R_w) \leq max(\sigma_1^2, \sigma_2^2)$ for all portfolios w.

   (3b) Suppose $\sigma_1 = \sigma_2$, and $\rho = 0$.

   - Solve for w^*, the portfolio with minimum return variance.
   - Compute $Var(R_{w^*})$.

   (3c) Suppose $\sigma_1 = \sigma_2$ (no assumptions about ρ)

   - Solve for w^*, the portfolio with minimum return variance.
   - Compute $Var(R_{w^*})$.
   - Graph the variance of the mininum-variance portfolio as a function of $\rho : -1 \leq \rho \leq 1$.

4. Consider a single-period analysis of $m > 2$ risky assets with

Returns:

$$\boldsymbol{R} = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} \text{ with mean} E[R] = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix},$$

and covariance

$$Cov[\boldsymbol{R}] = \Sigma = \begin{bmatrix} \Sigma_{1,1} & \cdots & \Sigma_{1,m} \\ \vdots & \ddots & \vdots \\ \Sigma_{m,1} & \cdots & \Sigma_{m,m} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho_{1,2}\sigma_1\sigma_2 & \cdots & \rho_{1,m}\sigma_1\sigma_m \\ \rho_{2,1}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2,m}\sigma_2\sigma_m \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{m,1}\sigma_m\sigma_2 & \rho_{m,2}\sigma_m\sigma_2 & \cdots & \sigma_m^2 \end{bmatrix}$$

where $\sigma_j = \sqrt{\Sigma_{j,j}}$, $j = 1, \ldots, m$, is the standard deviation of asset j's return, $j = 1, \ldots, m$

and $\rho_{i,j} = \Sigma_{i,j}/\sqrt{\Sigma_{i,i}\Sigma_{j,j}}$ is the return correlation between assets i and j, for $1 \leq i, j \leq m$.

A portfolio $\boldsymbol{w} = (w_1, w_2, \ldots, w_m)^T$ identified by its investment weights in the assets, has return:

$$R_w = \sum_{j=1}^m w_j R_j = \boldsymbol{w}^T \boldsymbol{R}.$$

Assume that w is fully invested ($\sum_{j=1}^m w_j = 1$) and that no short sales are allowed ($w_j \geq 0$, for all j).

4(a) Prove that $Var(R_w) \leq max(\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2)$ for all portfolios w.

4(b) Suppose $\sigma_1 = \sigma_2 = \cdots = \sigma_m$, and $\rho = 0$.

- Solve for w^*, the portfolio with minimum return variance.
- Compute $Var(R_{w^*})$ and express it as a function of m, the number of assets.
- What is the limit of $Var(R_{w^*})$ as $m \to \infty$.

4(c) Suppose $\sigma_1 = \sigma_2 = \cdots = \sigma_m$ and $\rho_{i,j} \equiv \rho$ for all $i \neq j$. The correlation matrix of the m-vector of returns is said to be an **equicorrelation** matrix because all assets have the same pairwise correlations.

- For the case of $m = 2$, there is no constraint on ρ except the usual one: $-1 \leq \rho \leq 1$.
  Prove generally that $\rho \geq -\frac{1}{m-1}$.
  Hint: $\mathbf{1}_m = (1, \ldots, 1)^T$ is an eigen-vector of the matrix Σ: confirm, compute the eigen-value, and apply constraints for Σ to be positive semi-definite.

- Solve for w^*, the portfolio with minimum return variance.
- Compute $Var(R_{w^*})$.
- Graph the variance of the mininum-variance portfolio as a function of $\rho : -\frac{1}{m-1} \leq \rho \leq 1$.
- What is the limit of $Var(R_{w^*})$ as $m \to \infty$?
- Compare this limit with that in (b) and comment on the ability to diversify away portfolio variability by adding additional (equi-correlated) assets to a portfolio.

5. Consider a single-period analysis of m risk assets as in problems 3 and 4.

5(a) Suppose $m = 2$, $\sigma_1 \neq \sigma_2$, and $\rho = \Sigma_{1,2}/\sqrt{\Sigma_{1,1}\Sigma_{2,2}} = 0$.

- Solve for w^*, the portfolio with minimum return variance.
- Compute $Var(R_{w^*})$.

5(b) Suppose $m > 2$, and $\Sigma = diag(\sigma_1^2, \ldots, \sigma_m^2)$, a diagonal matrix with no constraints on the asset variances (σ_j^2) and zero correlations between assets ($\rho_{i,j} = \Sigma_{i,j}/\sqrt{\Sigma_{i,j}\Sigma_{i,j}} \equiv 0$).

- Solve for w^*, the portfolio with minimum return variance.
- Compute $Var(R_{w^*})$.
- Express $Var(R_{w^*})$ as a function of m and $\tilde{\sigma}^2 = \frac{1}{\frac{1}{m}\sum_{j=1}^{m}\sigma_j^{-2}}$

5(c) Suppose $m > 2$, and no constraints on the positive definite matrix Σ.

- Solve for w^*, the portfolio with minimum return variance.
- Compute $Var(R_{w^*})$.