18.S096 Problem Set 4 Fall 2013
Time Series
Due Date: 10/15/2013

1. **Covariance Stationary AR(2) Processes**

   Suppose the discrete-time stochastic process $\{X_t\}$ follows a second-order auto-regressive process $AR(2)$:

   $$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \eta_t,$$

   where $\{\eta_t\}$ is $WN(0, \sigma^2)$, with $\sigma^2 > 0$, and ϕ_0, ϕ_1, ϕ_2, are the parameters of the autoregression.

   (a) If $\{X_t\}$ is covariance stationary with finite expectation $\mu = E[X_t]$ show that

   $$\mu = \frac{\phi_0}{1-\phi_1-\phi_2}$$

   (b) For the autocovariance function

   $$\gamma(k) = Cov[X_t, X_{t-k}],$$

   show that

   $$\gamma(k) = \phi_1\gamma(k-1) + \phi_2\gamma(k-2), \text{ for } k = 1, 2, \ldots$$

   (c) For the autocorrelation function

   $$\rho_k = corr[X_t, X_{t-k}],$$

   show that

   $$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, \text{ for } k = 1, 2, \ldots$$

   (d) **Yule-Walker Equations**

   Define the two linear equations for ϕ_1, ϕ_2 in terms of ρ_1, ρ_2 given by $k = 1, 2$ in (c):

   $$\begin{aligned} \rho_1 &= \phi_1\rho_0 + \phi_2\rho_{-1} \\ \rho_2 &= \phi_1\rho_1 + \phi_2\rho_0 \end{aligned}$$

   Using the facts that $\rho_0 = 1$, and $\rho_k = \rho_{-k}$, this gives

   $$\begin{aligned} \rho_1 &= \phi_1 + \phi_2\rho_1 \\ \rho_2 &= \phi_1\rho_1 + \phi_2 \end{aligned}$$

   which is equivalent to:

   $$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

   Solve for ϕ_1 and ϕ_2 in terms of ρ_1, and ρ_2.

(e) Solve for ρ_1, and ρ_2 in terms of ϕ_1 and ϕ_2.

(f) Derive complete formulas for ρ_k, $k > 2$. Hint: Solve this part by referring to the answer to problem 4(b) below.

2. **Difference equations associated with an $AR(p)$ process.**

An $AR(p)$ process,

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \eta_t$$

where $\{\eta_t\}$ is $WN(0, \sigma^2)$, with $\sigma^2 > 0$, and auto-regression parameters $\phi_0, \ldots, \phi_p$, can be written using the polynomial-lag operator

$$\phi(L) = (I - \phi_1 L - \phi_2 L^2 \cdots - \phi_p L^p)$$

as follows:

$$\phi(L) X_t = \eta_t.$$

Consider the homogeneous difference equations:

$$\phi(L) g(t) = 0.$$

for a discrete-time function $g(t)$.

Such equations arise in $AR(p)$ models when analyzing auto-covariance, auto-correlation, and prediction functions.

(a) If $\mu = E[X_t]$, show that

$$\mu = \frac{\phi_0}{1 - \sum_{i=1}^{p} \phi_i}$$

So, $\mu = 0$, if and only if $\phi_0 = 0$. It is common practice to 'de-mean' a time series which has the result of eliminating the autoregression parameter associated with the mean, i.e., ϕ_0.

(b) For the auto-covariance function,

$$\gamma(t) = Cov(X_t, X_0),\ t = 0, 1, \ldots,$$

prove that

$$\phi(L)\gamma(t) = 0, \text{ for all } t;$$

(c) For the auto-correlation function,

$$\rho(t) = Corr(X_t, X_0),\ t = 0, 1, \ldots,$$

prove that

$$\phi(L)\rho(t) = 0, \text{ for all } t.$$

(d) Suppose that the process is observed up to time t^*, so the values $(x_0, x_1, \ldots, x_{t^*})$ are known. Let $g_{t^*}(h)$, for $h = 1, 2, \ldots$, be forecasts of the process:

$$g_{t^*}(h) = E[X_{t^*+h} \mid (X_{t^*}, \ldots, X_1, X_0) = (x_{t^*}, \ldots, x_1, x_0)]$$

With the notation $\hat{x}_{t^*}(h) = g_{t^*}(h)$, for $h > 0$ and defining $\hat{x}_{t^*}(h) = x_{t^*+h}$, for $h \leq 0$, show that

$$\begin{aligned} \hat{x}_{t^*}(1) &= \phi_0 + \textstyle\sum_{i=1}^p \phi_i \hat{x}_{t^*}(1-i) \\ \hat{x}_{t^*}(2) &= \phi_0 + \textstyle\sum_{i=1}^p \phi_i \hat{x}_{t^*}(2-i) \\ &\vdots \\ \hat{x}_{t^*}(h) &= \phi_0 + \textstyle\sum_{i=1}^p \phi_i \hat{x}_{t^*}(h-i), \quad \text{for any } h > 0. \end{aligned}$$

(e) For the de-meaned process in (c) (i.e., $\phi_0 = 0$), show that the forecast function $g_{t^*}(t)$ satisfies

$$\phi(L) g_{t^*}(t) = 0, \text{ for } t = 1, 2, \ldots$$

3. **Solutions to Difference Equations Associated with $AR(p)$ Processes**

In the previous problem, the autocovariance, autocorrelation and forecast functions satisfy the difference equations:

$$\phi(L) g(t) = 0, \text{ for } t = 0, 1, 2, \ldots \tag{3.1}$$

(a) As a possible solution, consider the function:

$$g(t) = Ce^{\lambda t}, \text{ for some constants } C \text{ and } \lambda.$$

Show that such a function $g()$ is a solution if $z = e^{-\lambda}$ is a root of the characteristic equation:

$$\phi(z) = 1 - \textstyle\sum_{i=1}^p \phi_i z^p = 0.$$

(b) More generally, consider the function:

$g(t) = CG^t$, for some constants C and G (this is the generalization of (a) where $G = e^{-\lambda}$ is constrained to be real and positive).

Show that such a function $g()$ is a solution if G^{-1} is a root of the characteristic equation.

(c) For an $AR(p)$ process, suppose the p roots of the characteristic equation $z_i = (G_i^{-1})$ are distinct. Show that for constants $C_1, C_2, \ldots, C_p$, the function

$$g(t) = C_1 G_1^t + C_2 G_2^t + \cdots + C_p G_p^t$$

satisfies

$$\phi(L) g(t) = 0.$$

(d) For an $AR(p)$ process, if all roots z_i are outside the complex unit circle, prove that any solution $g()$ given in (b) is bounded.

Since the autocovariance function is a solution, this condition is necessary and sufficient for the autocovariance to be bounded (covariance stationary).

4. Consider an $AR(2)$ process.

   (a) Solve for the 2 roots of the characteristic equation, z_1 and z_2. Detail the conditions under which z_1 and z_2 are

   - Distinct and real.
   - Coincident and real.
   - Complex and conjugates of each other (i.e., $z_1 = a + ib$ and $z_2 = a - ib$, for two real constants a and b, and the imaginary $i = \sqrt{-1}$.

   (b) Prove that the autocorrelation function satisfies:

   $$\rho_k = C_1 G_1^k + C_2 G_2^k$$

   where $G_1 = z_1^{-1}$, and $G_2 = z_2^{-1}$

   Using equations for two values of k, solve for C_1 and C_2 when z_1 and z_2 are distinct, and show that

   $$\rho_k = \frac{G_1(1-G_2)^2 G_1^k - G_2(1-G_1^2)G_2^k}{(G_1-G_2)(1+G_1G_2)}$$

   (c) When distinct and real, show that the autocovariance function decreases geometrically/exponentially in magnitude.

   (d) In (c), under what conditions on ϕ_1 does the autocovariance/correlation function remain positive; and under what conditions does it alternate in sign.

   (e) **Optional:** When the two roots are complex conjugates, define d and f_0 so that:

   $$G_1 = de^{i2\pi f_0}, \text{ and } G_2 = de^{-i2\pi f_0}.$$

   Using part (b), it can be shown that

   $$\rho_k = [sgn(\phi_1)]^k |d|^k \frac{sin(2\pi f_0 k + F)}{sin(F)}$$

   where

   - The damping factor d is

     $$|d| = \sqrt{-\phi_2}$$

   - The frequency f_0 is such that

     $$2\pi f_0 = arccos\left(\frac{|\phi_1|}{2\sqrt{-\phi_2}}\right)$$

- The phase F is such that

$$tanF = \left(\frac{1+d^2}{1-d^2}\right) tan(2\pi f_0)$$

5. **Moving Average Process MA(1)**

Suppose the discrete stochastic process $\{X_t\}$ follows a MA(1) model:

$$\begin{aligned} X_t &= \eta_t + \theta_1 \eta_{t-1} \\ &= (1+\theta_1 L)\eta_t, \quad t = 1, 2, \ldots \quad (\eta_0 = 0) \end{aligned}$$

where $\{\eta_t\}$ is $WN(0, \sigma^2)$.

(a) Derive the auto-correlation function (ACF) of X_t

(b) For the following two model cases, solve for the first-order auto-correlation:

- $\theta_1 = 0.5$
- $\theta_1 = 2.0$

(c) Using the formula for $\rho_1 = corr(X_t, X_{t-1})$, in (a), solve for the MA process parameter θ_1 in terms of ρ_1.

Is the solution unique?

(d) An $MA(1)$ process is invertible if the process equation can be inverted, i.e., the process $\{X_t\}$ satisfies:

$$(1-\theta_1 L)^{-1} X_t = \eta_t$$

For each model case in (b), determine whether the process is invertible, and if so, provide an explicit expression for the model process as an (infinite-order) autoregression.

6. **Autoregressive Moving Average Process:** $ARMA(1,1)$

Suppose the discrete stochastic process $\{X_t\}$ follows a covariance stationary ARMA(1,1) model:

$$\begin{aligned} X_t - \phi_1 X_{t-1} &= \phi_0 + \eta_t + \theta_1 \eta_{t-1} \\ (1-\phi_1 L)X_t &= \phi_0 + (1+\theta_1 L)\eta_t, \quad t = 1, 2, \ldots \quad (\eta_0 = 0) \end{aligned}$$

where $\{\eta_t\}$ is $WN(0, \sigma^2)$.

(a) Prove that

$$\mu = E[X_t] = \frac{\phi_0}{1-\phi_1}$$

(b) Prove that

$$\sigma_X^2 = Var(X_t) = \gamma_0 = \frac{\sigma^2[1+\theta_1^2+2\phi_1\theta_1]}{1-\phi_1^2} = \sigma^2\left[1+\frac{(\theta_1+\phi_1)^2}{1-\phi_1^2}\right]$$

(c) Prove that the auto-correlation function (ACF) of the covariance stationary $ARMA(1,1)$ process is given by the following recursions:

$$\begin{aligned} \rho_1 &= \phi_1 + \frac{\theta_1 \sigma^2}{\gamma_0} \\ \rho_k &= \phi_1 \rho_{k-1}, \quad k > 1 \end{aligned}$$

(d) Compare the ACF of the $ARMA(1,1)$ process to that of the $AR(1)$ process with the same parameters ϕ_0, ϕ_1.

Note that

- Both decline geometrically/exponentially in magnitude by the factor ϕ_1 from the second time-step on.
- If $\phi_1 > 0$, both processes have an ACF that is always positive.
- If $\phi_1 < 0$, the ACFs alternate in sign from the second time-step on.

What pattern in the ACF function of an $ARMA(1,1)$ model is not possible with an $AR(1)$ model? Suppose an economic index time series follows such an $ARMA(1,1)$ process. What behavior would it exhibit?