Riemann Zeta and Random Matrices

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Topics

- Spacing of Riemann Zeta zeros
- Spacing of eigenvalues of random Gaussian matrices
- Numerical solution of the Painlevé V nonlinear differential equation
- Eigenvalues of the Prolate matrix

Riemann Zeta Zeros

- $\zeta\left(\frac{1}{2} + i\gamma_n\right) = 0, \qquad 0 < \gamma_1 < \gamma_2 < \ldots$
- $\gamma_n = n^{\text{th}}$ zero on the critical line
- Odlyzko: γ_{N+n}, $N = 0, 10^6, 10^{12}, 10^{18}, n = 1, 2, \ldots, 10^5$
- Normalize:

$$\tilde{\gamma}_n = \frac{\gamma_n}{\text{av spacing near } \gamma_n} = \gamma_n \cdot \left[\frac{\log \gamma_n / 2\pi}{2\pi}\right]$$

- Histogram of consecutive spacings $\tilde{\gamma}_{n+1} - \tilde{\gamma}_n$

zetadistr.m

```
delta=[];
for filename={'zeros3.txt','zeros4.txt','zeros5.txt'}
  a=textread(filename{1},'%s',1,'whitespace','\n');
  offset=eval(a{1}(find(a{1}>='0' & a{1}<='9')));
  gamma=textread(filename{1},'%n','headerlines',9);
  delta=[delta;diff(gamma)/2/pi.*log((gamma(1:end-1)+offset)/2/pi)];
end

dx=0.05;
x=0:dx:5.0;
H=histc(delta,x);
H=H(1:end-1);
H=H/sum(H)/dx;
xmid=(x(1:end-1)+x(2:end))/2;

bar(xmid,H)
grid on
```

Riemann Zeta Zeros

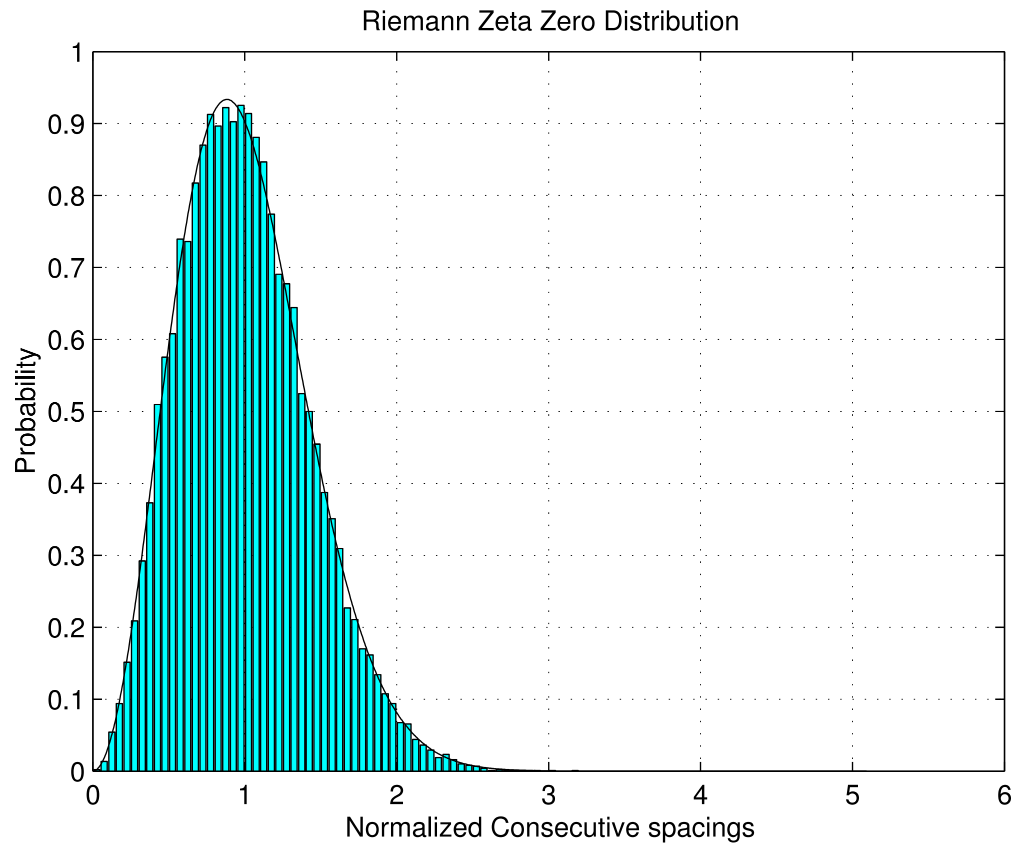

Figure 1: Probability distribution of consecutive spacings of Riemann Zeta zeros ($30,000$ zeros, $n \approx 10^{12}, 10^{21}, 10^{22}$)

Random Matrix Eigenvalues

- Hermitian $N \times N$ matrix A, diagonal elements x_{jj}, upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$, independent, zero-mean Gaussians, and

$$\begin{cases} \mathrm{Var}(x_{jj}) = 2, & 1 \le j \le N \\ \mathrm{Var}(u_{jk}) = \mathrm{Var}(v_{jk}) = 1, & 1 \le j < k \le N \end{cases}$$

- MATLAB:

```
A=randn(N)+i*randn(N);
A=(A+A')/sqrt(2);
```

- Normalized spacings of eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$:

$$\delta_n' = \frac{\lambda_{n+1} - \lambda_n}{4\pi} \sqrt{8N - \lambda_n^2}, \qquad n \approx N/2$$

Faster Method

- Real, symmetric, tridiagonal matrix with the same eigenvalues as the previous matrix for $\beta = 2$ (Dumitriu, Edelman):

$$H_\beta \sim \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$$

- Complexity n^2 instead of n^3.

- MATLAB: Interface to LAPACK function `DSTEQR` implemented in mex-file `trideig`

eigdistr.m

```
N=1000;
nrep=1000;

ds=[];
for ii=1:nrep
  % A=randn(N)+i*randn(N);
  % A=(A+A')/sqrt(2);
  % l=eig(A);

  l=trideig(sqrt(2)*randn(N,1),sqrt(chi2rnd((N-1:-1:1)'*2)));

  d=diff(l(N/4:3*N/4))/4/pi.*sqrt(8*N-l(N/4:3*N/4-1).^2);
  ds=[ds;d];
end

% Histogram as before
```

Random Matrix Eigenvalues

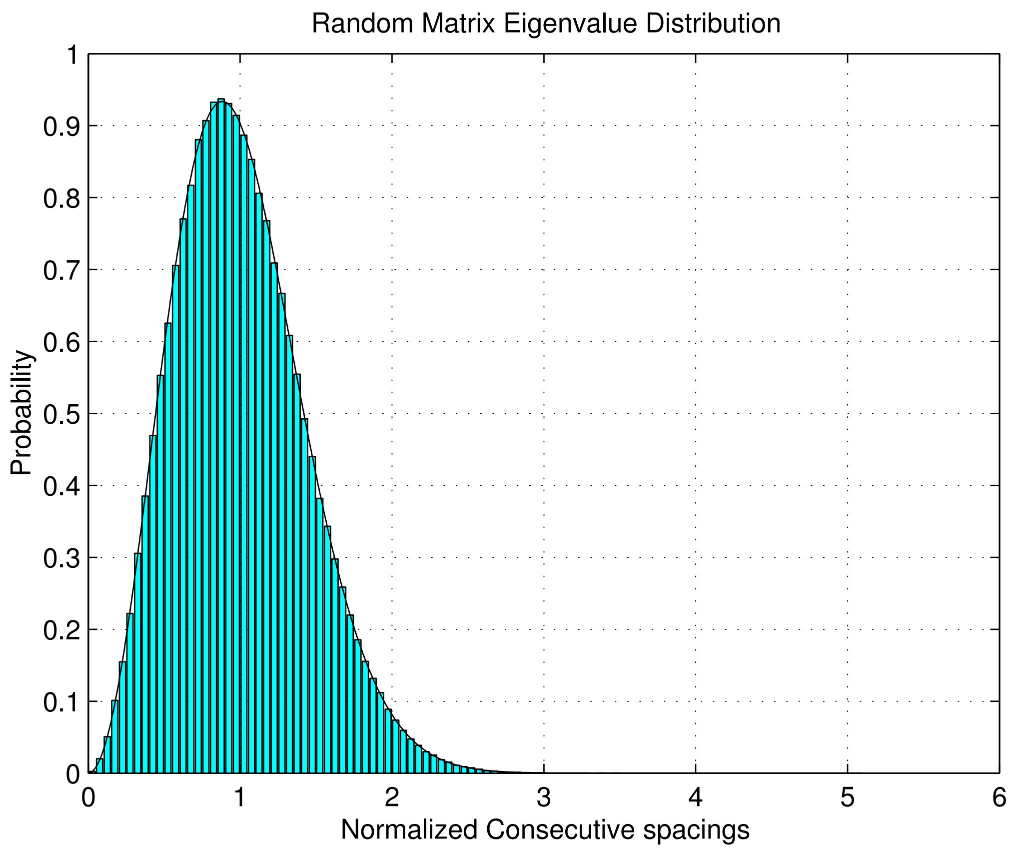

Figure 2: Probability distribution of consecutive spacings of random matrix eigenvalues (1000 repetitions, $N = 1000$)

Differential Equation for Distributions

- Probability distribution $p(s)$ is given by

$$p(s) = \frac{d^2}{ds^2} E(s)$$

where

$$E(s) = \exp\left(\int_0^{\pi s} \frac{\sigma(t)}{t} dt\right)$$

and $\sigma(t)$ satisfies the Painlevé V differential equation:

$$(t\sigma'')^2 + 4(t\sigma' - \sigma)\left(t\sigma' - \sigma + (\sigma')^2\right) = 0$$

with the boundary condition

$$\sigma(t) \approx -\frac{t}{\pi} - \left(\frac{t}{\pi}\right)^2, \qquad \text{as } t \to 0^+$$

Numerical Solution

- Write as 1^{st} order system:

$$\frac{d}{dt}\begin{pmatrix}\sigma\\ \sigma'\end{pmatrix} = \begin{pmatrix}\sigma'\\ -\frac{2}{t}\sqrt{(\sigma - t\sigma')\left(t\sigma' - \sigma + (\sigma')^2\right)}\end{pmatrix}$$

- Solve as initial-value problem starting at $t = t_0 =$small positive number

- Initial values (boundary conditions):

$$\begin{cases} \sigma(t_0) &= -\frac{t_0}{\pi} - \left(\frac{t_0}{\pi}\right)^2 \\ \sigma'(t_0) &= -\frac{1}{\pi} - \frac{2t_0}{\pi} \end{cases}$$

- Explicit ODE solver (RK4)

Post-processing

- $E(s) = \exp\left(\int_0^{\pi s} \frac{\sigma(t)}{t} dt\right)$ could be computed using high-order quadrature
- Convenient trick: Add variable $I(t)$ and equation $\frac{d}{dt} I = \frac{\sigma}{t}$ to ODE system, and let the solver do the integration
- $p(s) = \frac{d^2}{ds^2} E(s)$ using numerical differentiation

desig.m

```
function dy=desig(t,y)

s=y(1); ds=y(2);
dy=[ds; -2/t*sqrt((s-t*ds)*(t*ds-s+ds^2)); s/t];
```

solvsig.m

```
t0=1e-12;
tn=16;
tspan=[t0,tn];
tspan=linspace(t0,tn,1000);
y0=[-t0/pi-(t0/pi)^2; -1/pi-2*t0/pi; 0];

opts=odeset('reltol',1e-13,'abstol',1e-14);
[t,y]=ode45(@desig,tspan,y0,opts);
```

solvp.m

```
solvsig
E=exp(y(:,3));
s=t/pi;
p=gradient(gradient(E,s),s);
```

Painlevé V

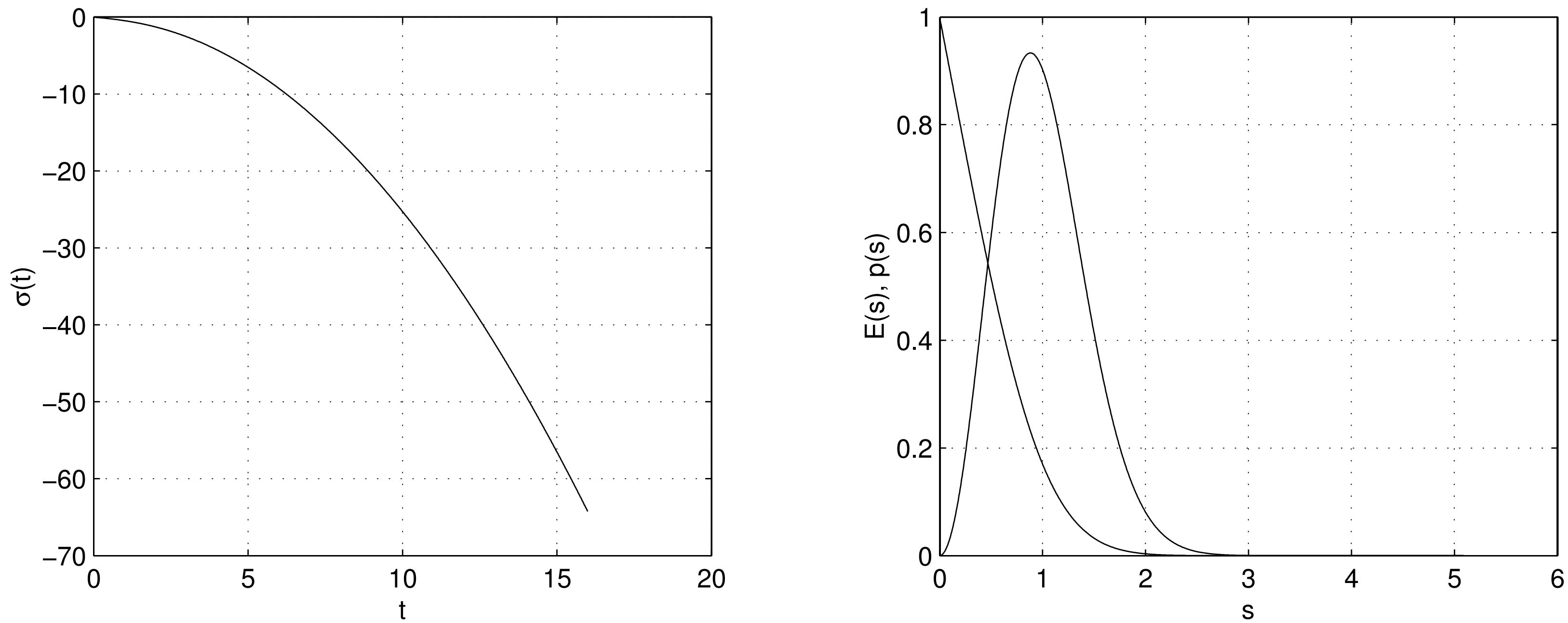

Figure 3: Painlevé V (left), $E(s)$ and $p(s)$ (right).

The Prolate Matrix

- $E(2t) = \prod_i (1 - \lambda_i)$ where λ_i are eigenvalues of the operator

$$f(y) \to \int_{-1}^{1} Q(x,y) f(y)\, dy, \qquad Q(x,y) = \frac{\sin((x-y)\pi t)}{(x-y)\pi}$$

- Infinite symmetric Prolate matrix:

$$A_\infty = \begin{pmatrix} a_0 & a_1 & \cdots \\ a_1 & a_0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

  with $a_0 = 2w$, $a_k = (\sin 2\pi wk)/\pi k$ for $k = 1, 2, \ldots$, and $0 < w < \frac{1}{2}$.

- Set $w = t/n$. The upper-left $n \times n$ submatrix A_n is then a discretization of $Q(x,y)$.

Faster Method?

- A_n commutes with the symmetric tridiagonal matrix (Slepian)

$$T_n = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

where

$$\begin{cases} \alpha_k &= \left(\frac{n+1}{2} - k\right)^2 \cos 2\pi w \\ \beta_k &= \frac{1}{2} k(n-k) \end{cases}$$

- Compute eigenvectors to T_n instead of A_n, and eigenvalues using dot products
- Still n^3

eigprolate.m

```
t=s/2;
E=zeros(size(t));
for ii=1:length(t)
  Q=gallery('prolate',N,t(ii)/N);
  E(ii)=prod(1-eig(Q));
end
```

richardson.m

```
solvp
E0=E;

Es=zeros(length(t),0);
for N=20*2.^(0:3)
  eigprolate
  Es=[Es,E];
end

for ii=1:3
  max(abs(Es-E0(:,ones(1,size(Es,2)))))
  Es=Es(:,2:end)+diff(Es,1,2)/(2^(ii+1)-1);
end
max(abs(Es-E0))
```

Accuracy

- Difference between Prolate solution $E(s)$ and Painlevé V solution $E_0(s)$:

$$\max_{0\leq s\leq 5} |E(s) - E_0(s)|$$

after 0, 1, 2, and 3 Richardson extrapolations:

N	Error 0	Error 1	Error 2	Error 3
20	0.2244			
40	0.0561	0.7701		
80	0.0140	0.0483	0.5486	
160	0.0035	0.0032	0.0323	2.2619
	$\cdot 10^{-3}$	$\cdot 10^{-7}$	$\cdot 10^{-8}$	$\cdot 10^{-11}$