SYMPLECTIC GEOMETRY, LECTURE 24

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1. Homeomorphism Classification of Simply Connected Compact 4-Manifolds

Theorem 1 (Freedman). *M_1, M_2 compact, simply connected, oriented smooth 4-manifolds are homeomorphic $\Leftrightarrow$ the intersection pairings $Q_i : H_2(M_i,\mathbb{Z}) \times H_2(M_i,\mathbb{Z}) \to \mathbb{Z}$ are isomorphic as integer quadratic forms (of $|\det| = 1$). It suffices to check that the following invariants coincide: $b_2 = \mathrm{rk} H_2(M), \sigma = b_2^+ - b_2^-$ (the signature), and $Q(x,x) \mod 2 \forall x$ (the parity).*

Example. For $M = \mathbb{CP}^2$, we have $Q_{\mathbb{CP}^2} = (1)$ on $H_2(\mathbb{CP}^2,\mathbb{Z}) = \mathbb{Z}[\mathbb{CP}^2]$, while $N = \overline{\mathbb{CP}^2}$ (with reversed orientation) has $Q_{\overline{\mathbb{CP}^2}} = (-1)$. By Mayer-Vietoris, one can see that $H_2(M\#N) = H_2(M) \oplus H_2(N)$ and $Q_{M\#N} = Q_M \oplus Q_N$. Applying this to m copies of $\mathbb{CP}^2$ and n copies of $\overline{\mathbb{CP}^2}$ gives the matrix $\begin{pmatrix} I_{m\times m} & \\ & -I_{n\times n} \end{pmatrix}$ which gives all the unimodular odd quadratic forms ($|\det| = 1$).

Example. $Q_{S^2\times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has $b_2^+ = b_2^- = 1$ like $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, but different parity.

Example. A $K3$ is a surface of degree 4 in $\mathbb{CP}^3$ (given, for instance, by $\{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$). We have $b_2 = 22, b_2^+ = 3, b_2^- = 19$, and $Q = 2.(-E_8) \oplus 3. \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where $(-E_8)$ is the matrix

$$(1) \qquad \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & 0 & 1 \\ & & & & 1 & -2 & 1 & 0 \\ & & & & 0 & 1 & -2 & 0 \\ & & & & 1 & 0 & 0 & -2 \end{pmatrix}$$

Theorem 2 (Donaldson). *In the even case, $Q = (2k).(\pm E_8) \oplus m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

Remark. The Rokhlin signature theorem ($16|\sigma$ in the even case) implies that the number of $\pm E_8$ summands is even.

Remark. The $\frac{11}{8}$ conjecture claims that the m in the theorem above satisfies $m \geq 3k$: it has been shown (Furuta, 1995) that $m \geq 2k$.

Remark. Smooth compact 4-manifolds may have infinitely many exotic smooth structures: $K3$ surfaces are known to have infinitely many smooth structures, as do the manifolds $\mathbb{CP}^2\#n.\overline{\mathbb{CP}^2}$ for $n \geq 3$.

2. Seiberg-Witten Invariants [J. Morgan], [Witten '94]

Let X^4 be a compact manifold, with Riemannian metric g and spin^c structure s. The goal of Seiberg-Witten theory is to assign a number to the pair (g, s) giving the number of "abelian supersymmetric magnetic monopoles" on the manifold.

Definition 1. *A* spin^c *structure is a rank 4 Hermitian complex vector bundle $S \to X$ along with a* Clifford multiplication *(unitary representation of a Clifford algebra) $\gamma : T^*X \times S \to S$ (i.e. $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2\langle u, v\rangle \mathrm{id}$ and $\gamma(u)^* = -\gamma(u)$).*

Example. For $\{e_i\}$ an orthonormal basis, $\gamma(e^i) \in U(S)$, since $\gamma(e^i)^2 = -1$, and $\gamma(e^i)\gamma(e^j) + \gamma(e^j)\gamma(e^i) = 0$.

We extend our Clifford multiplication to

$$\gamma : \bigwedge{}^*(T^*X) \times S \to S, \gamma(e^{i_1} \wedge \cdots \wedge e^{i_p}) = \gamma(e^{i_1}) \cdots \gamma(e^{i_p}) \tag{2}$$

for (e^i) orthonormal. Applying this to the volume form gives $\gamma(\mathrm{vol})^2 = (\gamma(e^1)\gamma(e^2)\gamma(e^3)\gamma(e^4))^2 = \mathrm{id}$ and thus a decomposition $S = S^+ \oplus S^-$, with the former having $\gamma(\mathrm{vol}) = -1$ and the latter $\gamma(\mathrm{vol}) = 1$. Moreover, $\gamma(e^i)$ maps $S^\pm$ to $S^\mp$.

Lemma 1. *We can represent complexified forms via* $\gamma : \wedge^* \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{End}(S^+ \oplus S^-)$. *More specifically, we have decompositions*

$$\begin{aligned} \wedge^{even} \otimes \mathbb{C} &\cong \mathrm{End}(S^+) \oplus \mathrm{End}(S^-) \\ \wedge^{odd} \otimes \mathbb{C} &\cong \mathrm{Hom}(S^+, S^-) \oplus \mathrm{Hom}(S^-, S^+) \end{aligned} \tag{3}$$

with $\gamma(*\alpha) = \gamma(\alpha)$ *on* S^+ *and* $-\gamma(\alpha)$ *on* S^- *for any* $\alpha \in \wedge^2$, *so*

$$\mathrm{End}(S^+) = \mathbb{C} \oplus (\wedge^2_+ \otimes \mathbb{C}), \mathrm{End}(S^-) = \mathbb{C} \oplus (\wedge^2_- \otimes \mathbb{C}) \tag{4}$$

Theorem 3. *Every compact* 4*-manifold admits* spin^c *structures classified up to* 2*-torsion by* $c = c_1(S^+) = c_1(S^-) = c_1(L) \in H^2(X, \mathbb{Z})$, *where* $L = \det\,(S^+) = \wedge^2 S^+ = \wedge^2 S^-$. *Moreover,* c *is a characteristic element, i.e.* $\langle c_1(L), \alpha\rangle \equiv Q(\alpha, \alpha) \mod 2$.

In particular, if $E \to X$ is a line bundle, the mapping $(S^+, S^-) \mapsto (S^+ \otimes E, S^- \otimes E)$ gives a twisting of the spin^c structure by any line bundle.

Proposition 1. *If* X *admits a* g*-orthogonal almost-complex structure* J, *then* $\exists$ *a* canonical spin^c *structure given by* $S^+ = \wedge^{0,0} \oplus \wedge^{0,2}, S^- = \wedge^{(0,1)}$ *with*

$$\forall u \in T^*X, \gamma(u) = \sqrt{2}[(u^{0,1} \wedge \cdot) - \iota_{(u^{1,0})^\#}(\cdot)] \tag{5}$$

Note that $L = \wedge^2 S^- = \wedge^2 S^+ = \wedge^{0,2}$ *is the anti-canonical bundle. All other spin structures are given by* $S^+ = E \oplus (\wedge^{0,2} \otimes E), S^- = \wedge^{0,1} \otimes E, \forall E \to X$ *a line bundle.*

3. Dirac Operator

Definition 2. *A* spin^c connection *on* $S^\pm$ *is a Hermitian connection* ∇^A *s.t.*

$$\nabla^A_v(\gamma(u)\phi) = \gamma(\nabla^{LC}_v u)\phi + \gamma(u)\nabla^A_v\phi \tag{6}$$

Proposition 2. *Any two* spin^c *connections differ by a* 1*-form on* X *of the type* $ia \otimes \mathrm{id}_{S^\pm}$, *and the induced connection* A *on* $L = \wedge^2 S^\pm$ *defines the* spin^c *connection uniquely.*

Definition 3. *Given a* spin^c *structure and a connection, the* Dirac operator *is*

$$D_A : \Gamma(S^\pm) \to \Gamma(S^\pm),\ D_A\psi = \sum_i \gamma(e^i)\nabla^A_{e_i}\psi \tag{7}$$

for $\{e_i\}$ *an orthonormal basis (though it is independent of choice of basis).*

Example. On a Kähler manifold, $S^+ = E \oplus \wedge^{0,2} \otimes E, S^- = \wedge^{0,1} \otimes E$, ∇^A corresponds to a unitary connection ∇^a on E, i.e. via $\nabla^A = \nabla^{LC} \otimes 1 + 1 \otimes \nabla^a$. Then $D_A = \sqrt{2}(\overline{\partial}_a + \overline{\partial}^*_a)$ and $D_A^2 = 2\overline{\Box}_a$.

Definition 4. *The* Seiberg-Witten equations *are the equations*

$$\begin{aligned} D_A\psi &= 0 \in \Gamma(S^-) \\ \gamma(F_A^+) &= (\psi^* \otimes \psi)_0 \in \Gamma(\mathrm{End}(S^+)) \end{aligned} \tag{8}$$

where A *is a Hermitian connection on* $L = \wedge^2 S^\pm$ *(corresponding to a* spin^c *connection on* $S^\pm$*),* $\psi \in \Gamma(S^+)$ *is a section,* $F_A^+ = \frac{1}{2}(F_A + *F_A) \in i\Omega^2_+$ *for* $F_A \in i\Omega^2$ *the curvature of* A, *and* $(\psi^* \otimes \psi)_0 = \psi^* \otimes \psi - \frac{1}{2}|\psi|^2$ *is the traceless part of* $\psi^* \otimes \psi$.