SYMPLECTIC GEOMETRY, LECTURE 23

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1. Branched Covers

Definition 1. *For (M, ω) a symplectic manifold, $p \in M$, a local diffeomorphism $\phi : U \to \mathbb{C}^n$ for $U \ni p$ is* ω-tame *if $(\phi_* \omega)(v, iv) > 0\ \forall v \neq 0 \in \mathbb{C}^n$. This is, $\phi^* J_0$ is ω-tame, i.e. complex lines in $\mathbb{C}^n$ give symplectic submanifolds in M.*

Definition 2. *A map $f : X^4 \to (Y^4, \omega_Y)$ from a compact, oriented manifold to a compact, symplectic manifold is a* symplectic branched covering *if $\forall p \in X, \exists U \ni p, V \ni f(p)$ coordinate neighborhoods (with $\phi : U \to \mathbb{C}^2$ an oriented diffeomorphism, $\psi : V \to \mathbb{C}^2$ an ω_Y-tame diffeomorphism) s.t. the right vertical map of the commutative diagram*

$$\begin{array}{ccc} X \supset U & \xrightarrow{\ \phi\ } & \mathbb{C}^2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \psi f \phi^{-1}} \\ Y \supset V & \xrightarrow[\ \psi\ ]{} & \mathbb{C}^2 \end{array} \tag{1}$$

is one of

1. $(u, v) \mapsto (z_1, z_2) = (u, v)$ *(local diffeomorphism),*
2. $(u, v) \mapsto (z_1, z_2) = (u^2, v)$ *(simple branching),*
3. $(u, v) \mapsto (z_1, z_2) = (u^3 - uv, v)$ *(cusp)*

Remark. Simple branching also makes sense in higher dimensions as $(x_1, \dots, x_n) \mapsto (x_1^2, x_2, \dots, x_n)$. Moreover, we could allow higher order branching, i.e. $(u, v) \mapsto (u^p, v)$ for $p > 2$, but this isn't generic.

Remark. The three models given above correspond to the 3 generic local models for holomorphic maps $\mathbb{C}^2 \to \mathbb{C}^2$.

Definition 3. *The* ramification curve *is the set $R \subset X$ s.t. $R = \{p \in X | df(p)$ not onto$\}$. The* branch (discriminant) curve *is $D = f(R) \subset Y$, i.e. $D = \{q \in Y | \# f^{-1}(q) < \deg f\}$.*

We can calculate these curves explicitly in local coordinates. For instance, in the case of simple branching, we have that $\text{Jac}(f) = \det\,(df) = \left| \begin{pmatrix} 2u & 0 \\ 0 & 1 \end{pmatrix} \right| = 2u$, so $R = \{u = 0\}, D = \{z_1 = 0\}$. In the case of a cusp, we have

$$\text{Jac}(f) = \det\,(df) = \left| \begin{pmatrix} 3u^2 - v & -u \\ 0 & 1 \end{pmatrix} \right| = 3u^2 - v \tag{2}$$

so $R = \{v = 3u^2\}$ and $D = \{27z_1^2 = 4z_2^3\}$. What happens at the cusp: $\forall p \in R, \text{Ker } df = \mathbb{C} \times \{0\} \subset T_p\mathbb{C}^2$, so $\text{Ker } df$ is transverse to TR at most points, but not at the cusp.

Lemma 1. *$R \subset X$ is a smooth, 2-dimensional submanifold, and $D \subset Y$ is a symplectic submanifold of Y immersed except at isolated points (corresponding to cusps). In local coordinates, D is a complex curve, so TD consists of complex lines and $\omega_Y|_{TD} > 0$.*

Note that the generic singularities of D consist of two types: complex cusps and transverse double points (with either orientation, i.e. $T_qY = T_1 \oplus T_2$ with agreeing or disagreeing orientations).

Proposition 1. *If $f : X^4 \to (Y^4, \omega_Y)$ is a symplectic branched covering, then X carries a symplectic form ω_X (canonical up to isotopy) s.t. $[\omega_X] = f^*[\omega_Y]$.*

Proof. Note that $f^*\omega_Y$ is a closed 2-form which is nondegenerate outside of R. $\forall p \in R, K_p = \text{Ker } df_p \subset T_pX$ is the kernel of $f^*\omega_Y$, and is a complex line in local coordinates (so it carries a natural orientation). We claim that $\exists\alpha$ an exact 2-form on X s.t. $\forall p \in R, \alpha|_{K_p} > 0$ is positive nondegenerate. Assuming this, we have that $\omega_X = f^*\omega_Y + \epsilon\alpha$ for $\epsilon > 0$ sufficient small is closed and nondegenerate, since

$$\omega_X \wedge \omega_X = f^*\omega_Y \wedge f^*\omega_Y + 2\epsilon f^*\omega \wedge \alpha + \epsilon^2\alpha \wedge \alpha \tag{3}$$

with the first term ≥ 0 everywhere and nondegenerate outside R, the second term positive on R (in local coordinates, $f^*\omega_Y = \frac{i\lambda}{2}(dv \wedge d\overline{v})$ for some $\lambda > 0$, and $\alpha|_{\mathbb{C}\times\{0\}} > 0$), and the third term negligible for small ϵ.

We are left to prove our claim. Fix $p \in R$, and choose local coordinates (u, v) on X of our model. Set $x = \text{Re}(u), y = \text{Im}(u)$, and $\alpha_p = d(\chi_1(|u|)\chi_2(|v|)xdy)$, where χ_1, χ_2 are cutoff functions chosen s.t. $\forall q \in R \cap \text{Supp}(\chi_2), \chi_1 \equiv 1$. In local coordinates, we have that

$$\forall(u, v) \in R \cap \text{Supp}(\chi_2), \alpha_p|_{K=\mathbb{C}\times\{0\}} = \chi_2(|v|)dx \wedge dy > 0 \tag{4}$$

and 0 outside $\text{Supp}(\chi_2)$. Covering R by small neighborhoods, and taking α to be the sum of these α_p gives the desired exact form.

Finally, to see that the choice of ω_X is canonical up to isotopy, note that the set of α's satisfying our claim (i.e. exact 2-forms s.t. $\alpha|_K > 0$ along R) is convex. That is, we can find an ϵ sufficiently small s.t., for two such forms α_1, α_2,

$$f^*\omega_Y + \epsilon\alpha_1,\ f^*\omega_Y + \epsilon\alpha_2,\ f^*\omega_Y + \epsilon(t\alpha_1 + (1-t)\alpha_2) \tag{5}$$

are all symplectic. □

There is a converse to this result. Let (X^4, ω) be a complex symplectic, $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{Z})$, $L \to X$ a line bundle s.t. $c_1(L) = \frac{1}{2\pi}[\omega]$, J a compatible almost-complex structure, etc. Recall that $L^{\otimes k}$ has many approximately-holomorphic sections: choosing three "good" sections, we obtain a map $f : X \to \mathbb{CP}^2$ which locally looks like one of our models. That is,

Theorem 1. *Every compact symplectic 4-manifold with integral $\frac{1}{2\pi}[\omega]$ can be realized as a symplectic branched cover of $\mathbb{C}P^2$.*

This f_k will look like the local models in coordinates which are ω-tame on X and ω_0-tame on $\mathbb{C}P^2$, and applying the proposition to f_k gives $[f_k^*\omega_0] = k[\omega]$ with ω_X isotopic to $k\omega$. Moreover, if k is large enough, then $\exists$ a preferred choice of $f_k : X \to \mathbb{C}P^2$ up to homotopy among symplectic branched covers.

Remark. If D is holomorphic, then we can lift J_0 to X, i.e. X is a Kähler manifold and f is holomorphic. Conversely, if X is not Kähler, then the singular symplectic curve $D \subset \mathbb{CP}^2$ is not isotopic to any holomorphic curve.