SYMPLECTIC GEOMETRY, LECTURE 21

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1. Counterexamples contd.

We continue our discussion of the Thurston manifold introduced last time. Recall that $M = \mathbb{R}^4/\Gamma$, where Γ is generated by the four maps

$$\begin{aligned} g_1 &: (x_1, x_2, x_3, x_4) \mapsto (x_1 + 1, x_2, x_3, x_4) \\ g_2 &: (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2 + 1, x_3 + x_4, x_4) \\ g_3 &: (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3 + 1, x_4) \\ g_4 &: (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4 + 1) \end{aligned} \tag{1}$$

We showed last time that M was symplectic.

Lemma 1. *$H_1(M, \mathbb{Z}) = \mathbb{Z}^3$.*

Proof. One way to see this is to note that $g_3 = [g_4, g_2]$, so $\mathrm{Ab}(\Gamma) = \Gamma/[\Gamma, \Gamma] = \Gamma/\langle g_3\rangle \cong \mathbb{Z}^3$. To see this another way, note that $\pi_1(M) = \Gamma$ is generated by the four loops $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ given by the coordinate axes in $\mathbb{R}^4$. Look at γ_4: this can be described as

$$\gamma_4 \sim \{(a_1, a_2, a_3, t), t \in [0, 1]\} \sim \{(a_1, a_2 - 1, a_3, t), t \in [0, 1]\} \sim \{(a_1, a_2, a_3 + t, t), t \in [0, 1]\} \tag{2}$$

implying that $[\gamma_4] = [\gamma_3] + [\gamma_4]$ and $[\gamma_3] = 0$ in $H_1(M)$ (so the space is generated by the images of the other three loops). □

2. Symplectic Fibrations

Let $f : M \to B$ be a locally trivial fibration, with generic fiber (F, ω_F) a symplectic manifold.

Definition 1. *f is* symplectic *if the structure group reduces to* $\mathrm{Symp}(F, \omega_F)$*, i.e. $\exists$ local trivializations $f : f^{-1}(U_i) \cong U_i \times F \to U_i$ s.t., over $U_i \cap U_j$, the change of trivialization is a symplectomorphism.*

Now, let $f : M \to B$ be a compact, locally trivial symplectic fibration with symplectic fiber (F, ω_F) and symplectic base (B, ω_B).

Theorem 1 (Thurston)**.** *If $\exists c \in H^2(M, \mathbb{R})$ s.t. $c|_F = [\omega_F] = H^2(F, \mathbb{R})$. Then $\forall k >> 0, \exists$ a symplectic form on M in the class $c + k.f^*[\omega_B]$ for which the fibers of f are symplectic submanifolds.*

Proof. Choose a closed 2-form η on M s.t. $[\eta] = c$, and a cover $\{U_i\}$ of B by contractible subsets with trivializations $\phi_i : f^{-1}(U_i) \to F \times U_i$ s.t. $\phi_i \circ \phi_j^{-1}$ are symplectomorphisms over $U_i \cap U_j$. Let $p_i = \mathrm{pr}_2 \circ \phi_i : f^{-1}(U_i) \to F$. Then, on $U_i \times F$, η and $p_i^*\omega_F$ are closed 2-forms, and

$$[\eta|_{f^{-1}(U_i)}] = [p_i^*\omega_F] \in H^2(f^{-1}(U_i), \mathbb{R}) \cong H^2(F, \mathbb{R}) \tag{3}$$

since $c|_F = [\omega_F]$. Thus, $\exists$ a 1-form α_i on $f^{-1}(U_i)$ s.t. $p_i^*\omega_F = \eta + d\alpha_i$. Now, let ρ_i be a partition of unity on B by smooth functions $\rho_i : B \to [0, 1], \mathrm{Supp}(\rho_i) \subset U_i$, and set $\tilde{\eta} = \eta + \sum d((\rho_i \circ f)\alpha_i)$. Then $\tilde{\eta}$ is closed, with $[\tilde{\eta}] = [\eta] = c$: moreover,

$$\tilde{\eta}|_{F_p = f^{-1}(p)} = \eta|_{F_p} + \sum_i (\rho_i \circ f) d\alpha_i|_{F_p} = \sum_i \rho_i(f(p))(\eta|_{F_p} + d\alpha_i|_{F_p}) = \omega_F \tag{4}$$

in the trivializations ϕ_i.

We have obtained a closed 2-form $\tilde{\eta}$ on M s.t. $[\tilde{\eta}] = c$ which is symplectic on the fibers. $\forall x \in M$, split $T_xM = V_x \oplus H_x$, where $V_x = \mathrm{Ker}\ df_x$ is the tangent space to the fiber and $H_x = \{v \in T_xM | \tilde{\eta}(v, v') = 0 \forall v' \in$

$V_x\}$. These two spaces are in direct sum since $\tilde{\eta}|_{V_x}$ is nondegenerate. $f^*\omega_B$ is nondegenerate on H_x because $df_x : H_x \xrightarrow{\sim} T_{f(x)}B$, so $\tilde{\eta} + kf^*\omega_B$ is nondegenerate on H_x for $k >> 0$ since nondegeneracy is an open condition (consider $f^*\omega_B + \frac{1}{k}\tilde{\eta}$). It is also nondegenerate on V_x since $(\tilde{\eta} + kf^*\omega_B)|_{V_x} = \tilde{\eta}|_{V_x}$. Thus, we obtain our desired symplectic form on M. □

Remark. Assume dim $F = 2$: then if F is orientable and the fibration is oriented, we always have a symplectic form ω_F, and the structure group always reduces to $\text{Symp}(F, \omega_F) = \text{Diff}^+_{\text{vol}}(F)$. The cohomological assumption in the theorem is equivalent to the statement that $[f^{-1}(\text{pt})] \neq 0 \in H_2(M, \mathbb{R})$ (for instance, it is true on the Kodaira-Thurston manifold).

We can generalize this to other settings.

Definition 2. *A* Lefschetz fibration *is a map $M^4 \to \Sigma^2$ between oriented manifolds with isolated critical points modeled in oriented coordinates on $\mathbb{C}^2 \to \mathbb{C}, (z_1, z_2) \to z_1^2 + z_2^2$ (so the central fibers is the union of two lines, and nearby fibers are smooth conics).*

Theorem 2 (Gompf, 1998)**.** *If $f : M^4 \to \Sigma^2$ is a Lefschetz fibration with $[F] \neq 0 \in H_2(M, \mathbb{R})$, then M carries a symplectic form s.t. the fibers are symplectic.*

Theorem 3 (Donaldson)**.** *For (M^4, ω) symplectic, after blowing up points in M, we get $\hat{M}$ which admits a Lefschetz fibration to S^2. Here, the blowup is locally given by $\hat{\mathbb{C}}^2 = \{(x, \ell) \in \mathbb{C}^2 \times \mathbb{CP}^1 | x \in \ell\}$.*

The idea of this theorem is to look at approximately holomorphic sections $s, s' \in C^\infty(L^{\otimes k})$ s.t. s/s' is an "approximately meromorphic" function and has nondegenerate critical points.

3. Symplectic Sums (Gompf 1994)

Definition 3. *A* symplectic sum *is a connected sum along a codimension* 2 *symplectic submanifold.*

Explicitly, for $Q^{2n-2} \subset (M^{2n}, \omega)$ a compact, symplectic submanifold, $NQ = (TQ)^\perp$ is a rank 2 symplectic vector bundle over Q. Putting a compactible complex structure on it gives $c_1(NQ) \in H^2(Q, \mathbb{Z})$. Assume NQ is trivial, so $c_1(NQ) = 0$ (i.e. it has a nonvanishing section).

Example. For $n = 2$, c_1 is precisely the degree of the line bundle, and deg $(NQ) = [Q] \cdot [Q]$ because the zeroes of a section of NQ are obtained by deforming Q to Q' and intersecting them.

Now, by the symplectic tubular neighborhood theorem, we have a neighborhood of Q in M symplectomorphic to $(Q \times D^2(\epsilon), \omega|_Q \oplus \omega_0)$. Idea: use exponential maps to identify $\phi : v(Q) \xrightarrow{\sim} Q \times D^2(\epsilon)$ s.t. $\phi_*\omega$ and $(\omega|_Q \oplus \omega_0)$ agree on $Q \times \{0\}$, and use local Moser to produce a local symplectomorphism to identify to two forms.