SYMPLECTIC GEOMETRY, LECTURE 20

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Recall from last time the statement of the following lemma: given L a holomorphic line bundle with curvature $-i\omega$,

Lemma 1. *$\forall s \in C^\infty(L^{\otimes k})$, $\exists \xi \in C^\infty(L^{\otimes k})$ st. $||\xi||_{L^2} \leq \frac{C}{\sqrt{k}} \left|\left|\overline{\partial} s\right|\right|_{L^2}$ and $s + \xi$ is holomorphic.*

Proof. For this, we use the Weitzenbock formula for

$$\overline{\square}_k = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \Omega^{0,1}(L^{\otimes k}) \circlearrowleft \tag{1}$$

where $\overline{\partial}$ is induced by ∇ on $L^{\otimes k}$. We fix $p \in M$ and work in a neighborhood with p identified with the origin, choosing a standard frame for $T_pM \cong \mathbb{C}^n$ with $e_i = \frac{\partial}{\partial z_i}$ an orthonormal frame of $T^{1,0}$, $e^i = dz_i$ the dual frame. Using parallel transport w.r.t. the Levi-Cevita connection in the radial directions, we still have these frames (though they are no longer given by coordinates). At the origin, moreover, we have $\nabla_{e_i} e_j = 0$. Now,

$$\overline{\partial}\alpha = \sum_i \overline{e^i} \wedge \nabla_{\overline{e_i}}\alpha, \ \overline{\partial}^*\alpha = -\sum_i i_{\overline{e_i}}(\nabla_{e_i}\alpha) \tag{2}$$

so at the origin

$$\overline{\square}_k\alpha = -\sum_{i,j} i_{\overline{e_i}}(\overline{e^j} \wedge \nabla_{e_i}\nabla_{\overline{e_j}}\alpha) - \sum_{i,j} \overline{e^j} \wedge (i_{\overline{e_i}}\nabla_{\overline{e_j}}\nabla_{e_i}\alpha) \tag{3}$$

Note that $\nabla_{\overline{e_j}}\nabla_{e_i}\alpha = \nabla_{e_i}\nabla_{\overline{e_j}}\alpha - R(e_i, \overline{e_j})\alpha$, where

$$R = R^{T^*M} \otimes \mathrm{id}_{L^k} + \mathrm{id}_{T^*M} \otimes R^{L^k} \tag{4}$$

is the curvature on $T^*M \otimes L^k$. Now, $i_{\overline{e^i}}\overline{e^j} \wedge \cdot$ maps $\overline{e^i} \mapsto 0$ and is the identity on other terms when $i = j$ and, when $i \neq j$, sends $\overline{e^i}$ to $-\overline{e^j}$ and other terms to 0. Similarly, $\overline{e^j} \wedge (i_{\overline{e_i}} \cdot)$ sends $\overline{e^i}$ to $\overline{e^j}$ and maps the other terms to zero. Thus,

$$\begin{aligned} \overline{\square}_k\alpha &= -\sum_i \nabla_{e_i}\nabla_{\overline{e_i}}\alpha + \sum_{i,j} \overline{e^j} \wedge i_{\overline{e_i}}(R(e_i, \overline{e_j})\alpha) \\ &= D\alpha + R\alpha + \sum_i \overline{e^i} \wedge i_{\overline{e_i}}(k\alpha) = D\alpha + R\alpha + k\alpha \end{aligned} \tag{5}$$

Here, D is a semipositive operator, as $\int_M \langle D\alpha, \alpha\rangle = \int_M \left|\overline{\partial}\alpha\right|^2 \geq 0$, while R has order 0 and is independent of k. Thus,

$$\int \langle \overline{\square}_k\alpha, \alpha\rangle \mathrm{vol}_0 = \int \langle D\alpha, \alpha\rangle + \int \langle R\alpha, \alpha\rangle + \int k\left|\alpha\right|^2 \geq 0 - C\left|\left|\alpha\right|\right|^2_{L^2} + k\left|\left|\alpha\right|\right|^2_{L^2} \tag{6}$$

for some constant C. If $k > C$, then Ker $\overline{\square}_k = 0$ and (by self-adjointness under L^2) Coker $\overline{\square}_k = 0$, so $\overline{\square}_k$ is invertible. Furthermore, the smallest eigenvalue of $\overline{\square}_k$ is $\geq k - C$, so $\overline{\square}_k$ admits an inverse G with norm $\leq \frac{1}{k-C} \leq \mathcal{O}(\frac{1}{k})$.

Finally, given $s \in C^\infty(L^k)$, let $\xi = -\overline{\partial}^* G\overline{\partial}s$.

(1) $s + \xi$ is holomorphic:

$$\overline{\partial}^\nabla(s + \xi) = \overline{\partial}s - \overline{\partial}\overline{\partial}^* G\overline{\partial}s = (\overline{\square}_k - \overline{\partial}\overline{\partial}^*)G\overline{\partial}s = \overline{\partial}^*\overline{\partial}G\overline{\partial}s \tag{7}$$

But Im $\overline{\partial} \cap$ Im $\overline{\partial}^* = \{0\}$, since $\overline{\partial}a = \overline{\partial}^*b \implies \left|\left|\overline{\partial}a\right|\right|^2_{L^2} = \langle\overline{\partial}a, \overline{\partial}^*b\rangle_{L^2} = \langle\overline{\partial}\overline{\partial}a, b\rangle_{L^2} = 0$. Thus, $\overline{\partial}(s + \xi) = 0$ as desired.

(2) $||\xi||^2_{L^2} =\leq \mathcal{O}(\frac{1}{k})\,||\overline{\partial}s||^2_{L^2}$:

$$(8)\quad \begin{aligned} \left|\left|\overline{\partial}^* G\overline{\partial}s\right|\right|^2_{L^2} &= \langle \overline{\partial}^* G\overline{\partial}s, \overline{\partial}^* G\overline{\partial}s\rangle_{L^2} = \langle \overline{\partial}\overline{\partial}^* G\overline{\partial}s, G\overline{\partial}s\rangle_{L^2} \\ &= \langle \overline{\partial}s, G\overline{\partial}s\rangle_{L^2} \leq ||G||\,||\overline{\partial}s||^2_{L^2} \leq \mathcal{O}(\frac{1}{k})\,||\overline{\partial}s||^2_{L^2} \end{aligned}$$

□

1. Counterexamples

We know now that K ahler $\implies$ complex and symplectic, while both imply the existence of an almost-complex structure, and the latter implies that the manifold is even-dimensional and orientable. In dimension 2, these are all the same: in dimension 4, all these inclusions are strict (even when restricting to compact manifolds).

Example.

- S^4 is even-dimensional and orientable, but not almost-complex: if it were, $c_1(TS^4, J) \in H^2(S^4, \mathbb{Z}) = 0$ would satisfy $c_1^2 \cdot [S^4] = 2c_2 - p_1 = 2\chi + 3\sigma$ (with χ the Euler characteristic and σ the signature), which is impossible. Similarly, $\mathbb{CP}^2\#\mathbb{CP}^2$ is not almost-complex:

$$(9)\quad c_1 = (a, b) \in H^2 \cong \mathbb{Z}^2 \implies c_1^2 \cdot [\mathbb{CP}^2\#\mathbb{CP}^2] = a^2 + b^2 \neq 2\chi + 3\sigma = 14$$

  which is again impossible.
- $\mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$ is almost-complex, but not symplectic or complex: Ehresman-Wu implies that $\exists J$ with $c_1 = c \in H^2(M, \mathbb{Z}) \Leftrightarrow c^2 \cdot [M] = 2\chi + 3\sigma$ and $\forall x \in H_2, \langle c, x\rangle \equiv Q(x, x) \mod 2$. In our case, $\chi = 5$ and $\sigma = 3$, so the calculation works out. By the Kodaira classification of surfaces, if it were complex it would be Kähler; by Taubes' (1995) theorem on Seiberg-Witten invariants, it is not symplectic.
- The Hopf surface $S^3 \times S^1 \cong (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$ is complex ($\mathbb{Z}$-action $(z_1, z_2) \mapsto (\lambda^n z_1, \lambda^n z_2)$ is holomorphic) but not symplectic ($H^2 = 0$).
- Not all symplectic manifolds have complex structure (compatible or otherwise). For the former case, we have examples of torus bundles over tori; for the latter case, we have the following theorem.

  Theorem 1 (Gompf 1994). *$\forall G$ finitely presented group, $\exists M^4$ compact, symplectic, but not complex with $\pi_1(M^4) \cong F$.*

  This construction is obtained by performing symplectic sums along codimension 2 symplectic submanifolds. Since

$$(10)\quad H_1(M, \mathbb{Z}) = \mathrm{Ab}(\pi_1(M)) = \mathrm{Ab}(G) = G/[G, G]$$

  M is not K ahler if this has odd rank (since $H^1 \cong H^{1,0} \oplus H^{0,1}$, with the two parts having the same rank). Using the Kodaira classification, one can arrange to obtain non-complex manifolds as well.
- The Kodaira-Thurston manifold $M = \mathbb{R}^4/\Gamma$, where Γ is the discrete group generated by

$$(11)\quad \begin{aligned} g_1 &: (x_1, x_2, x_3, x_4) \mapsto (x_1 + 1, x_2, x_3, x_4) \\ g_2 &: (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2 + 1, x_3 + x_4, x_4) \\ g_3 &: (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3 + 1, x_4) \\ g_4 &: (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4 + 1) \end{aligned}$$

  is complex and symplectic, but not Kähler. Note that $\Gamma \subset \mathrm{Symp}(\mathbb{R}^4, \omega_0)$ (obvious for the three translations, while $g_2^*\omega_0 = dx_1 \wedge d(x_2 + 1) + d(x_3 + x_4) \wedge dx_4 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ as desired), so M is symplectic. M is also a symplectic T^2 bundle over T^2, with the base given by x_1, x_2 and the fiber by x_3, x_4 (with the bundle trivial along the x_1 direction, nontrivial along the x_2 direction with monodromy $(x_3, x_4) \mapsto (x_3 + x_4, x_4)$).