SYMPLECTIC GEOMETRY, LECTURE 18

Prof. Denis Auroux

Let (M, ω, J) be a compact Kähler manifold, $[\omega] \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})$. Then we can find a line bundle $L \to M$ with first Chern class $c_1(L) = [\omega]$. Choose a Hermitian metric on L along with a Hermitian connection ∇ with $R^\nabla = -2\pi i \omega$. More explicitly, starting with any hermitian connection ∇, R^∇ is a closed imaginary 2-form: in a trivialization, $\nabla = d + A$, so $R^\nabla = dA + [A, A] = dA$. Thus,

$$[R^\nabla] = -2\pi i c_1(L) = -2\pi i[\omega] \implies \exists a \in \Omega^1(M) \text{ s.t. } R^\nabla = -2\pi i \omega + i da \tag{1}$$

Letting $\nabla' = \nabla - ia$, we find that $R' = R - ida = -2\pi i \omega$.

Next, recall that $\nabla^{0,1}$ defines a holomorphic structure on L iff $(R^\nabla)^{0,2} = 0$. Since ω is a $(1,1)$-form and $R^\nabla = -2\pi i \omega$, we get a holomorphic line bundle structure for L. We will furthermore see that $L^{\otimes k}$ has "enough holomorphic sections", i.e. the number of such sections $\to \infty$. Given this, consider a basis of holomorphic sections $s_0, \dots, s_N \in H^0(L)$ (or $H^0(L^{\otimes k})$). Assume that, $\forall p \in M, \exists s \in H^0(L)$ s.t. $s(p) \neq 0$. Then we can define a map

$$f : M \to \mathbb{CP}^n, p \to [s_0(x) : \cdots : s_N(x)] \tag{2}$$

More intrinsically, we obtain a map

$$M \to \mathbb{P}(H^0(L)^*), p \mapsto H_p = \{s \in H^0(L) | s(p) = 0\} \subset H^0(L) \tag{3}$$

Here, H_p is the kernel of the linear form given by evaluation at p, well-defined up to scaling.

Definition 1. *L is* very ample *if $f : M \to \mathbb{P}(H^0(L)^*)$ is a well-defined embedding, and* ample *if $L^{\otimes k}$ is very ample for some k.*

We can reformulate this using the Kodaira embedding theorem:

Theorem 1 (Kodaira)**.** *A holomorphic line bundle is ample $\Leftrightarrow$ it has a holomorphic connection whose curvature is a Kähler form.*

The traditional proof of the Kodaira embedding theorem requires the Kodaira vanishing theorem. Instead, we will prove this using Donaldson's argument. For simplicity, replace ω by $\frac{\omega}{2\pi}$, so $[\frac{\omega}{2\pi}] = c_1(L)$. We will explicitly construct holomorphic sections of $L^{\otimes k}$ for all $k >> 0$.

- First, fix $p \in M$, and choose local Darboux coordinates s.t. $\omega = \frac{i}{2} \sum dz_j \wedge d\overline{z_j}$ and $J = J_0 + \mathcal{O}(|z|)$ (we can't assume that J is the natural complex structure, because that would imply the Kähler metric was flat).
- Next, choose a unitary trivialization of $L^{\otimes k}$, so that ∇ corresponds to

$$d + iA_0 = d + \frac{k}{4} \sum z_j d\overline{z_j} - \overline{z_j} dz_j \tag{4}$$

To see that we can choose A in this way, note that, in any trivialization, $\nabla = d + iA$, so $-ik\omega = R = idA$. We have

$$idA_0 = \frac{k}{4} \sum dz_j \wedge d\overline{z_j} = -ik\omega_0 = idA \tag{5}$$

Thus, $A - A_0$ is closed and locally exact. Moreover, changing the trivialization by $f = e^{i\phi} \in C^\infty(U, U(1))$ changes the connection 1-form to $A' = A + d\phi$. Thus a suitable change of trivialization ensures that the connection form becomes iA_0.

Remark. Baby model: assume $J = J_0$ in our coordinates (so that the Kähler metric is flat), and consider $s(z) = \exp(-\frac{k}{4}|z|^2)$: this function arises from considering the curvature

(6) $$R^{1,1} = \partial^\nabla \overline{\partial}^\nabla + \overline{\partial}^\nabla \partial^\nabla = \overline{\partial}\partial \log|\sigma|^2$$

for σ a holomorphic section. We claim that s is holomorphic w.r.t. ∇. To see this, note that

(7) $$\nabla s = ds + iA_0 s = (-\frac{k}{4}\sum z_j d\overline{z_j} + \overline{z_j}dz_j)s + (\frac{k}{4}\sum z_j d\overline{z_j} - \overline{z_j}dz_j)s = \frac{-k}{2}(\sum \overline{z_j}dz_j)s$$

so $\overline{\partial}^\nabla s = 0$ as desired.

- In our case,

(8) $$J = J_0 + \mathcal{O}(|z|) \implies \left|\nabla s^{0,1}\right| = \mathcal{O}(|z|\cdot|\nabla s|) = \mathcal{O}(k|z|^2 \cdot |s|)$$

while

(9) $$|\nabla s| = \mathcal{O}(k|z||s|) \implies \frac{\sup\left|\nabla s^{0,1}\right|}{\sup|\nabla s|} = \mathcal{O}(\frac{1}{\sqrt{k}})$$

We say that s is "approximately holomorphic".

Definition 2. *A family of sections $s_k \in C^\infty(L^{\otimes k})$ is* uniformly bounded *if it satisfies the uniform bounds*

(10) $$\sup_{x\in M}|\nabla^r s_k|_g \le C_r k^{\frac{r}{2}}$$

and approximately holomorphic *if*

(11) $$\sup_{x\in M}\left|\nabla^{r-1}\overline{\partial}s_k\right|_g \le C_r k^{\frac{r-1}{2}}$$

for all r. Furthermore, s_k is uniformly concentrated *at p if $\exists$ a polynomial P and a constant $\lambda > 0$ s.t.*

(12) $$\forall x \in M, \left|\frac{1}{k^{t/2}}\nabla^t s(x)\right| \le P(\sqrt{k}d(p,x))\exp(-\lambda k \mathrm{dist}(p,x)^2)$$

for $t \in \{0,\ldots,r\}$.

Proposition 1. *If (M,ω) is a compact symplectic manifold with a compatible almost complex structure, then $\exists$ a family of sections $(\sigma_{k,p})_{k>>0,p\in M}$ which are uniformly bounded, approximately holomorphic, uniformly concentrated, and $|\sigma_{k,p}| \ge c > 0$ over $B(p,\frac{1}{\sqrt{k}})$.*

In the Kähler case, we also have the following approximation theorem.

Proposition 2. *Given a family of sections $\{\sigma_{k,p}\}$ as above, $\exists\{\tilde{\sigma}_{k,p}\}$ holomorphic s.t.*

(13) $$\sup(k^{r/2}|\nabla^r\sigma_{k,p} - \nabla^r\tilde{\sigma}_{k,p}|) \le Ce^{-\lambda k/3}$$

That is, any estimate you make via σ can also be applied to $\tilde{\sigma}$, so you can assume that your approximately holomorphic sections are holomorphic and obtain the desired embedding. To use these sections to prove Kodaira embedding, note that $\forall p \in M, \exists s \in H^0(L^{\otimes k})$ s.t. $s(p) \neq 0$ since $|\tilde{\sigma}_{k,p}(p)| \approx 1$ (that is, $L^{\otimes k}$ is base point free). Moreover, given $p \neq q \in M, \exists s, s' \in H^0(L^{\otimes k})$ s.t. $|s(p)| > |s(q)|$ and $|s'(p)| < |s'(q)|$: e.g., if p, q are distant by more than $k^{-\frac{1}{2}}$ we can take $s = \tilde{\sigma}_{k,p}$ and $s' = \tilde{\sigma}_{k,q}$ (that is, our sections separate points). Finally, at every point p, $\forall v \in T_pM, \exists\sigma_1,\sigma_2 \in H^0(L^{\otimes k})$ s.t. $d_v(\frac{\sigma_1}{\sigma_2}) \neq 0$ (that is, our sections separate tangent vectors). This is done by choosing a local holomorphic coordinate so that $v = \mathrm{Re}\frac{\partial}{\partial z_1}$ and perturbing $z_1\sigma_{k,p}$ to a holomorphic section; setting $\sigma_2 = \tilde{\sigma}_{k,p}$ gives the desired nonzero derivative.