SYMPLECTIC GEOMETRY, LECTURE 16

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Recall that we were in the midst of elliptic operator analysis of the Laplace-deRham operator $\Delta = (d + d^*)^2$. We claimed that Δ was an elliptic operator, i.e. it has an invertible symbol $\sigma(\xi) = -|\xi|^2 \,\mathrm{id}$. We stated that a differential operator $L : C^\infty(E) \to C^\infty(F)$ of order k extends to a map $L_s : W^s(E) \to W^{s-k}(F)$.

Definition 1. *For $L : \Gamma(E) \to \Gamma(F)$ a differential operator, $P : \Gamma(F) \to \Gamma(E)$ is called a* parametrix *(or* pseudoinverse*) if $L \circ P - \mathrm{id}_E$ and $P \circ L - \mathrm{id}_F$ are smoothing operators, i.e. they extend continuously to $W^s(E) \to W^{s+1}(E)$.*

The following results can be found in Wells' book.

Theorem 1. *Every elliptic operator has a pseudoinverse.*

Corollary 1. *$\xi \in W^s(E), L$ elliptic, $L\xi \in C^\infty(F) \implies \xi C^\infty(E)$.*

Theorem 2. *L elliptic $\implies L_s$ is Fredholm, i.e.* Ker L_s, Coker L_s *are finite dimensional,* Im L_s *is closed, and* Ker L_s = Ker $L \subset C^\infty(E)$.

Theorem 3. *L elliptic, $\tau \in (\mathrm{Ker}\ L^*)^\perp = \mathrm{Im}\ L \subset C^\infty(F) \implies \exists! \xi \in C^\infty(E)$ s.t. $L\xi = \tau$ and $\xi \perp \mathrm{Ker}\ L$.*

Theorem 4. *L elliptic, self-adjoint $\implies \exists H_L, G_L : C^\infty(E) \to C^\infty(E)$ s.t.*

(1) *H_L maps $C^\infty(E) \to \mathrm{Ker}\ (L)$,*
(2) *$L \circ G_L = G_L \circ L = \mathrm{id} - H_L$,*
(3) *G_L, H_L extend to bounded operators $W^s \to W^s$, and*
(4) *$C^\infty(E) = \mathrm{Ker}\ L \oplus_{\perp L^2} \mathrm{Im}\ (L \circ G_L)$.*

We now return to the case of $\Delta = (d + d^*)^2$ on a compact manifold.

Corollary 2. *$\exists G : \Omega^k \to \Omega^k$ and $H : \Omega^k \to \mathcal{H}^k = \mathrm{Ker}\ \Delta$ s.t. $G\Delta = \Delta G = \mathrm{id} - H$ and $\mathrm{Im}\ (G\Delta) = (\mathcal{H}^k)^\perp$.*

Corollary 3. *$\Omega^k = \mathcal{H}^k \oplus_{\perp L^2} \mathrm{Im}\ d \oplus_{\perp L^2} \mathrm{Im}\ d^*$.*

Remark. Every $\alpha \in \Omega^k$ decomposes as $\alpha = H\alpha + d(d^* G\alpha) + d^*(dG\alpha)$.

Using this decomposition, we immediately obtain the theorem

Theorem 5 (Hodge)**.** *For M a compact, oriented Riemannian manifold, every cohomology class has a unique harmonic representative.*

From now on, M is a compact, Kähler manifold, with the Hodge $*$ operator on $\Omega^*(M)$ extended $\mathbb{C}$-linearly to $\mathbb{C}$-valued forms.

Proposition 1. *$*$ maps $\bigwedge^{p,q} \to \bigwedge^{n-q,n-p}$.*

Proof. Consider the standard orthonormal basis of $V = T_x^* M$ given by $\{x_1, y_1, \ldots, x_n, y_n\}$ with $Jx_j = y_j$ and $z_j = x_j + iy_j$ giving the basis for $\bigwedge^{1,0}$. Now, write any form α as a linear combination of

$$\alpha_{A,B,C} = \prod_{j \in A} z_j \wedge \prod_{j \in B} \overline{z_j} \wedge \prod_{j \in C} z_j \wedge \overline{z_j} \tag{1}$$

where $A, B, C \subset \{1, \ldots, n\}$ are disjoint subsets. That is, A is the set of indices which contribute purely holomorphic terms of α, B is the set of indices which contribute purely anti-holomorphic terms to α, and C is the set of indices which contribute both. One can show that

$$*(\alpha_{A,B,C}) = i^{a-b} (-1)^{\frac{1}{2}k(k+1)+c} (-2i)^{k-n} \alpha_{A,B,C'} \tag{2}$$

where $C' = \{1, \ldots, n\} \setminus (A \cup B \cup C), a = |A|, b = |B|, c = |C|$, and $k = \deg\ \alpha = a + b + 2c$. By this, $(p, q) = (a + c, b + c)$-forms map to $(a + (n - a - b - c), b + (n - a - b - c)) = (n - q, n - p)$-forms as desired. $\square$

Let $L : \Omega^{p,q} \to \Omega^{p+1,q+1}$ be the map $\alpha \mapsto \omega \wedge \alpha, L^* : \Omega^{p,q} \to \Omega^{p-1,q-1}$ the adjoint map $\alpha \mapsto (-1)^{p+q} * L *$. Furthermore, set $d_C = J^{-1} dJ = (-1)^{k+1} JdJ$, with adjoint $d_C^* = J^{-1} d^* J = (-1)^{k+1} Jd^* J$. On functions, we have that

$$d_c f = -Jdf = -J(\partial f + \overline{\partial} f) = -i\partial f + i\overline{\partial} f = -i(\partial - \overline{\partial})f \tag{3}$$

which extends to higher forms as well. Thus, $dd_C = -i(\partial + \overline{\partial})(\partial - \overline{\partial}) = 2i\partial\overline{\partial}$.

Lemma 1. *For X Kähler, $[L, d] = 0, [L^*, d^*] = 0$, $[L, d^*] = d_C, [L^*, d] = -d_C^*$.*

Proof. The first part follows from $d(\alpha \wedge \omega) = d\alpha \wedge \omega$. For the second, see Wells, theorem 4.8. □

Proposition 2. $\Delta_C = J^{-1} \Delta J = d_C d_C^* + d_C^* d_C = \Delta$

Proof. By J-invariance of ω, we have that $[L, J] = [L^*, J] = 0$. Using the above identities, we have that $[L^*, d_C] = d^*$, so

$$\Delta = dd^* + d^* d = d[L^*, d_C] + [L^*, d_C]d = dL^* d_C - dd_C L^* + L^* d_C d - d_C L^* d \tag{4}$$

Conjugating by J simply swaps terms, since $dd_C = -d_C d$. □

Let

$$\begin{aligned} \overline{\partial^*} &= - * \partial * : \Omega^{p,q} \to \Omega^{p,q-1} \\ \partial^* &= - * \overline{\partial} * : \Omega^{p,q} \to \Omega^{p-1,q} \end{aligned} \tag{5}$$

so $d^* = \partial^* + \overline{\partial}^*$.

Lemma 2. *$\overline{\partial^*}$ is L^2-adjoint to $\overline{\partial}$, and ∂^* is L^2-adjoint to ∂.*

For $\phi, \psi \in \Omega^k(M, \mathbb{C})$, we have the natural scalar product

$$\langle \phi, \psi \rangle_{L^2} = \int_M \phi \wedge * \overline{\psi} \tag{6}$$

Under this, the various $\Omega^{p,q}$ are orthogonal because if $\phi \in \Omega^{p,q}, \psi \in \Omega^{p',q'}, (p, q) \neq (p', q')$, then $\phi \wedge *\overline{\psi}$ is of type

$$(n + (p - p'), n + (q - q')) \neq (n, n) \tag{7}$$

Finally, define the operators

$$\Box = \partial\partial^* + \partial^*\partial, \overline{\Box} = \overline{\partial\partial^*} + \overline{\partial^*\partial} : \Omega^{p,q} \to \Omega^{p,q} \tag{8}$$

Theorem 6. *For M compact, Kähler,*

$$H_{\overline{\partial}}^{p,q}(M) = \mathcal{H}_{\overline{\Box}}^{p,q} = \text{Ker } \overline{\Box} \tag{9}$$

The proof that each $\overline{\partial}$-cohomology class contains a unique $\overline{\Box}$-harmonic form is similar to that of the Hodge theorem in the Riemannian case.

Theorem 7. $\Delta = 2\Box = 2\overline{\Box}$.

Proof. By the first lemma, $d^* d_c = d^*[L, d^*] = d^* L d^* = -[L, d^*]d^* = -d_C d^*$. Moreover, $d_c = -i(\partial - \overline{\partial})$, so $\overline{\partial} = \frac{1}{2}(d - \text{i}d_c)$ and $\overline{\partial}^* = \frac{1}{2}(d^* + id_c^*)$. Thus,

$$\begin{aligned} 4\overline{\Box} &= (d - \text{i}d_c)(d^* + \text{i}d_c^*) + (d^* + \text{i}d_c^*)(d - \text{i}d_c) \\ &= (dd^* + d^* d) + (d_c d_c^* + d_c^* d_c) + i(dd_c^* + d_c^* d) - i(d_c d^* + d^* d_c) \\ &= \Delta + \Delta_c + 0 + 0 = 2\Delta \end{aligned} \tag{10}$$

□

Corollary 4. *Δ maps $\Omega^{p,q}$ to itself and*

$$H_{dR}^k(M, \mathbb{C}) = \mathcal{H}_\Delta^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q} = \bigoplus_{p,q} H_{\overline{\partial}}^{p,q}(M) \tag{11}$$