SYMPLECTIC GEOMETRY, LECTURE 15

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1. Hodge Theory

Theorem 1 (Hodge)**.** *For M a compact Kähler manifold, the deRham and Dolbeault cohomologies are related by $H^k_{dR}(M, \mathbb{C}) = \bigoplus_{p,q} H^{p,q}_{\bar\partial}(M)$, with $H^{p,q} \cong \overline{H^{q,p}}$.*

Before we discuss this theorem, we need to go over Hodge theory for a compact, oriented Riemannian manifold (M, g).

Definition 1. *For V an oriented Euclidean vector space, the* Hodge $*$ operator *is the linear map $\bigwedge^k V \to \bigwedge^{n-k} V$ which, for any oriented orthonormal basis $e_1, \dots, e_n$, maps $e_1 \wedge \cdots \wedge e_k \mapsto e_{k+1} \wedge \cdots \wedge e_n$.*

Example. For any V, $*(1) = e_1 \wedge \cdots \wedge e_n$, and $** = (-1)^{k(n-k)}$.

Applying this to $T^*_x M$, we obtain a map on forms.

Remark. Note that,

$$\forall \alpha, \beta \in \Omega^k, \alpha \wedge *\beta = \langle \alpha, \beta \rangle.\mathrm{vol} \tag{1}$$

Definition 2. *The* codifferential *is the map*

$$d^* = (-1)^{n(k-1)+1} * d* : \Omega^k(M) \to \Omega^{k-1}(M) \tag{2}$$

Proposition 1. *d^* is the L^2 formal adjoint to the deRham operator d, i.e. on a compact closed Riemannian manifold, $\forall \alpha \in \Omega^k, \beta \in \Omega^{k+1}$, we have*

$$\langle d\alpha, \beta \rangle_{L^2} = \int_M \langle d\alpha, \beta \rangle d\mathrm{vol} = \langle \alpha, d^*\beta \rangle_{L^2} \tag{3}$$

Proof. This follows from

$$\begin{aligned} \int_M \langle d\alpha, \beta \rangle d\mathrm{vol} &= \int_M d\alpha \wedge *\beta = \int_M d(\alpha \wedge *\beta) - (-1)^k \int_M \alpha \wedge d * \beta \\ &= (-1)^{k+1} \int_M \alpha \wedge d * \beta = (-1)^{k+1} \int_M \alpha \wedge *(*d * \beta)(-1)^{k(n-k)} \\ &= (-1)^{kn+1} \int_M \langle \alpha, *d * \beta \rangle d\mathrm{vol} \end{aligned} \tag{4}$$

□

Example. For $\mathbb{R}^n$ with the standard metric,

$$\alpha = \sum_{I \subset \{1,\dots,n\}} \alpha_I dx_I \implies d\alpha = \sum_j dx_j \wedge \frac{\partial \alpha}{\partial x_j} \text{ and } d^*\alpha = -\sum_j i_{\frac{\partial}{\partial x_j}} \frac{\partial \alpha}{\partial x_j} \tag{5}$$

Definition 3. *The* Laplacian *is $\Delta = dd^* + d^*d : \Omega^k \to \Omega^k$.*

Note that, on $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M), \Delta = (d + d^*)^2$. By the adjointness of d and d^*, we see that Δ is a self-adjoint, second order differential operator, i.e. $\langle \Delta\alpha, \beta \rangle_{L^2} = \langle \alpha, \Delta\beta \rangle_{L^2}$. Moreover,

$$\langle \Delta\alpha, \alpha \rangle_{L^2} = \langle dd^*\alpha, \alpha \rangle_{L^2} + \langle d^*d\alpha, \alpha \rangle_{L^2} = ||d^*\alpha||^2 + ||d\alpha||^2 \geq 0 \tag{6}$$

so $\Delta\alpha = 0 \Leftrightarrow \alpha$ is closed and co-closed.

Definition 4. *The space of* harmonic forms *is the set* $\mathcal{H}^k = \{\alpha \in \Omega^k | \Delta\alpha = 0\}$.

We have a natural map $\mathcal{H}^k \to H^k, \alpha \mapsto [\alpha]$.

Theorem 2 (Hodge)**.** *For M a compact, oriented Riemannian manifold, every cohomology class has a unique harmonic representative, i.e. $\mathcal{H}^k \cong H^k$, and $\Omega^k(M) = \mathcal{H}^k \oplus_{L^2} d(\Omega^{k-1}) \oplus_{L^2} d^*(\Omega^{k+1})$.*

Remark. Clearly $\mathcal{H}^k + d(\Omega^{k-1}) \subset \text{Ker } d = (\text{Im } d^*)^\perp$ and $\mathcal{H}^k + d^*(\Omega^{k+1}) \subset \text{Ker } d^* = (\text{Im } d)^\perp$, implying that the map $\mathcal{H}^k \to H^k$ is injective. Surjectivity (i.e. existence of harmonic representatives) is more difficult and requires elliptic theory.

Definition 5. *A* differential operator of order k *is a linear map $L : \Gamma(E) \to \Gamma(F)$ s.t., locally in coordinates,*

$$L(s) = \sum_{|\alpha| \le k} A_\alpha \frac{\partial^{|\alpha|} s}{\partial x^\alpha} \tag{7}$$

where each A_α is a C^∞ function with values in matrices, i.e. a local section of $\text{Hom}(E, F)$*. The* symbol *of L is the map*

$$\sigma_k : T_x^* M \ni \xi \mapsto \sum_{|\alpha| = k} A_\alpha(x) \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \in \text{Hom}(E_x, F_x) \tag{8}$$

L is elliptic *if for every nonzero ξ, $\sigma(\xi)$ is an isomorphism.*

Example. For instance, in local coordinates, the symbol of the Laplacian is given by $\sigma(\xi) = -|\sigma|^2 \cdot \text{id}$.

Now, let L be a differential operator of order k: it extends from $L : C^\infty(E) \to C^\infty(F)$ to $L_s : W^s(E) \to W^{s-k}(F)$.

Definition 6. *For $L : \Gamma(E) \to \Gamma(F)$ a differential operator, $P : \Gamma(F) \to \Gamma(E)$ is called a* parametrix *(or* pseudoinverse*) if $L \circ P - \text{id}_E$ and $P \circ L - \text{id}_F$ are smoothing operators, i.e. they extend continuously to $W^s(E) \to W^{s+1}(E)$.*

Using Rellich's lemma on embedding of Sobolev spaces, we find that

Theorem 3. *Every elliptic operator has a pseudoinverse.*

Corollary 1. *$\xi \in W^s(E)$, L is elliplic, and $L\xi \in C^\infty(F) \implies \xi \in C^\infty(E)$.*

Proof. Let P be a parametrix. Let $S = P \circ L - I$, so

$$\xi = P \circ L\xi - S\xi \tag{9}$$

Since the former part lies in $C^\infty(E)$ and the latter in $W^{s+1}(E)$, we have that $\xi \in W^{s+1}(E)$. Iterating, $\xi \in C^\infty(E)$. □