SYMPLECTIC GEOMETRY, LECTURE 13

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1. Integrability of Almost-Complex Structures

Recall the following:

Definition 1. *The* Nijenhuis tensor *is the form*

$$N_J(u,v) = [Ju, Jv] - J[u, Jv] - J[Ju, v] - [u, v] \tag{1}$$

Proposition 1. $N(u, v) = -8\mathrm{Re}([u^{1,0}, v^{1,0}]^{0,1})$.

Proof. $[u^{1,0}, v^{1,0}] = \frac{1}{4}[u - iJu, v - iJv] = \frac{1}{4}([u, v] - i[Ju, v] - i[u, Jv] - [Ju, Jv])$. Taking the real part of the $(0, 1)$ component gives the desired expression. □

Corollary 1. $N = 0$ *globally* $\Leftrightarrow [T^{1,0}, T^{1,0}] \subset T^{1,0}$*, i.e. the Lie bracket preserves the splitting* $T^{1,0} \oplus T^{0,1}$.

Proposition 2. N *is a tensor, i.e. in depends only on the values of* u, v.

Note also that N is by definition skew-symmetric an J-antilinear. In fact, N can be taken as a complex map $\bigwedge^2(TM, J) \to (TM, -J)$. Thus, if dim ${}_{\mathbb{R}}M = 2, N = 0$, since $N(u, Ju) = -JN(u, u) = 0$.

Definition 2. *An almost-complex structure J is a* complex structure *if it is* integrable*, i.e. if* $\exists$ *local holomorphic coordinates s.t.* $(M, J) \cong (\mathbb{C}^n, i)$ *locally.*

Proposition 3. *If J is a complex structure,* $N = 0$.

Proof. This follows from the fact that, on $T^{1,0}\mathbb{C}^n, [\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}] = 0$. □

Theorem 1 (Newlander-Nirenberg)**.** $N \equiv 0 \Leftrightarrow J$ *is integrable.*

Proof. Sketch: producing holomorphic coordinates is equivalent to giving a frame on the tangent bundle of the form $\{\frac{\partial}{\partial z_i}\}$, which is the same as finding a basis $\{e_i\}$ of $T^{1,0}$ s.t. $[e_i, e_j] = 0$. □

This does not make the problem of determining whether a manifold has some complex structure trivial: for instance, it is currently unknown whether S^6 has an integrable complex structure.

We can extend our tensor to differential forms to obtain alternate ways to determine integrability.

Proposition 4. *The dual map* $N^* : \bigwedge^{0,1} T^*M \to \bigwedge^{2,0} T^*M$ *is precisely the map* $N^*\alpha = (d\alpha)^{(2,0)}$.

Proof. For $\alpha \in \Omega^{0,1}$, we have a decomposition $d\alpha = \partial\alpha + \overline{\partial}\alpha + (d\alpha)^{(2,0)} \in \Omega^{1,1} \oplus \Omega^{0,2} \oplus \Omega^{2,0}$. Moreover,

$$\begin{aligned} d\alpha^{(2,0)}(u, v) &= d\alpha^{(2,0)}(u^{1,0}, v^{1,0}) = d\alpha(u^{1,0}, v^{1,0}) \\ &= u^{1,0} \cdot \alpha(v^{1,0}) - v^{1,0} \cdot \alpha(u^{1,0}) - \alpha([u^{1,0}, v^{1,0}]) \end{aligned} \tag{2}$$

The first two terms of the latter expression vanish, implying that $d\alpha(u^{1,0}, v^{1,0}) = 8\alpha(N(u, v))$. □

Similarly, for $\beta \in \Omega^{1,0}$, we have $\overline{N}^*\beta = (d\beta)^{(0,2)}$. Note that, for f a function, $df = \partial f + \overline{\partial} f$, so

$$ddf = d(\partial f) + d(\overline{\partial} f) = (\partial\partial f + \overline{\partial}\partial f + \overline{N}^*\partial f) + (N^*\overline{\partial} f + \partial\overline{\partial} f + \overline{\partial}\overline{\partial} f) \tag{3}$$

so $\overline{\partial}^2 f = -\overline{N}^*\partial f$. If f is holomorphic, $\overline{\partial} f = 0 \implies \overline{\partial\partial} f = 0 \implies \overline{N}^*\partial f = 0$. Therefore, if there exist $z_i : M \to \mathbb{C}$ holomorphic functions s.t. ∂z_i generate $T^*M^{1,0}$, then $N = 0$ and $\overline{\partial}^2 = 0$.

Theorem 2 (Newlander-Nirenberg)**.** *J is integrable* $\Leftrightarrow N \equiv 0 \Leftrightarrow [T^{1,0}, T^{1,0}] \subset T^{1,0} \Leftrightarrow d = \partial + \overline{\partial} \Leftrightarrow \overline{\partial}^2 = 0$ *on forms.*

Finally, we return to the case of M a symplectic manifold with compatible a.c.s. J and induced metric g. Denote by ∇ the Levi-Civita connection given by g. In this case, J is integrable $\Leftrightarrow \nabla(Jv) = J\nabla(v) \Leftrightarrow \nabla J = 0 \Leftrightarrow \nabla(T^{1,0}) \subset T^{1,0}$.

Definition 3. *A symplectic manifold* (M, ω, J) *is* Kähler *if* J *is integrable and compatible with* ω*. That is,* (M, J) *is a complex manifold,* ω *is a closed, positive, real, nondegenerate* $(1,1)$*-form (i.e.* $\omega(Ju, Ju) = \omega(u, v)$*).*

Example. $(\mathbb{C}^n, \omega_0, i)$ is Kähler.

Example. Any Riemann surface (oriented with area form) is Kähler.

Example. The complex projective space $\mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\}/(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ is Kähler. The points are given as homogeneous coordinates $[z_0 : \cdots : z_n]$, with coordinate charts

$$\mathbb{C}^n \cong U_i = \{z_i \neq 0\} = \{[\frac{z_0}{z_i} : \cdots : 1 : \cdots \frac{z_n}{z_i}]\} \tag{4}$$

and coordinate changes (WLOG on $U_0 \cap U_1$) given by $[1 : z_1 : \cdots : z_n] \mapsto [\frac{1}{z_1} : 1 : \frac{z_2}{z_1} : \cdots : \frac{z_n}{z_1}]$. Note that $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \cong S^2$: more generally,

$$\mathbb{C}P^n = \{[1 : z_1 : \cdots z_n]\} \sqcup \{[0 : z_1 : \cdots z_n] | z_i \neq 0 \text{ for some } i\} = \mathbb{C}^n \cup \mathbb{C}P^{n-1} \tag{5}$$

so we can construct the spaces inductively from cells in dimension $2i, i \in \{0, \ldots, n\}$.

We claim that $\mathbb{C}P^n$ has a symplectic structure compatible with the complex structure given above.