SYMPLECTIC GEOMETRY, LECTURE 12

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1. Existence of Almost-Complex Structures

Let (M, ω) be a symplectic manifold. If J is a compatible almost-complex structure, we obtain invariants $c_j(TM, J) \in H^{2j}(M, \mathbb{Z})$ of the deformation equivalence class of (M, ω).

Remark. There exist 4-manifolds (M^4, ω_1), (M^4, ω_2) s.t. $c_1(TM, \omega_1) \neq c_1(TM, \omega_2)$.

We can use this to obtain an obstruction to the existence of an almost-complex structure on a 4-manifold: note that we have two Chern classes $c_1(TM, J) \in H^2(M, \mathbb{Z})$ and $c_2(TM, J) = e(TM) \in H^4(M, \mathbb{Z}) \cong \mathbb{Z}$ if M^4 is closed, compact. Then the class

$$(1 + c_1 + c_2)(1 - c_1 + c_2) - 1 = -c_1^2 + 2c_2 = c_2(TM \oplus \overline{TM}, J \oplus \overline{J}) = c_2(TM \otimes_{\mathbb{R}} \mathbb{C}, i) \tag{1}$$

is independent of J.

More generally, for E a real vector space with complex structure J, we have an equivalence $(E \otimes_{\mathbb{R}} \mathbb{C}, i) \cong E \oplus \overline{E} = (E, J) \oplus (E, -J)$. Indeed, J extends $\mathbb{C}$-linearly to an almost complex structure $J_{\mathbb{C}}$ which is diagonalizable with eigenvalues $\pm i$. Applying this to vector bundles, we obtain the *Pontrjagin classes*

$$p_1(TM) = -c_2(TM \otimes_{\mathbb{R}} \mathbb{C}) \in H^4(M, \mathbb{Z}) \cong \mathbb{Z} \tag{2}$$

for a 4-manifold M.

Theorem 1. *$p_1(TM) \cdot [M] = 3\sigma(M)$, where $\sigma(M)$ is the signature of M (the difference between the number of positive and negative eigenvalues of the intersection product $Q : H_2(M) \otimes H_2(M) \to \mathbb{Z}, [A] \otimes [B] \mapsto [A \cap B]$ dual to the cup product on H^2).*

Corollary 1. $c_1^2 \cdot [M] = 2\chi(M) + 3\sigma(M)$.

Remark. Under the map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}/2\mathbb{Z})$, the Chern class $c_1(TM, J)$ gets sent to the *Stiefel-Whitney class* $w_2(TM)$. This means that

$$c_1(TM) \cdot [A] \equiv Q([A], [A]) \mod 2 \ \forall [A] \in H_2(M, \mathbb{Z}) \tag{3}$$

Theorem 2. *$\exists$ an almost complex structure J on M^k s.t. $\alpha = c_1(TM, J) \in H^2(M, \mathbb{Z})$ iff α satisfies*

$$\alpha^2 \cdot [M] = 2\chi + 3\sigma \text{ and } \alpha \cdot [A] \equiv Q([A], [A]) \mod 2 \ \forall [A] \in H_2(M, \mathbb{Z}) \tag{4}$$

Examples:

- On S^4, if J were an almost complex structure, then $c_1(TS^4, J) \in H^2(S^4) = 0.$. However, $\chi(S^4) = 2$ and $\sigma(S^4) = 0$, so $2 \cdot 2 + 3 \cdot 0$ cannot be c_1^2, and thus there is no almost complex structure.
- On $\mathbb{CP}^2$, we have $H_2(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$ generated by $[\mathbb{CP}^1]$ with intersection product $Q([\mathbb{CP}^1], [\mathbb{CP}^1]) = 1$ (the number of lines in the intersection of two planes in $\mathbb{C}^3$. By Mayer-Vietoris, $H_2(\mathbb{CP}^2 \# \mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}^2$ has intersection product $Q = I_{2\times 2} \implies \sigma = 2$ and Euler characteristic $\chi = 4$. Now, assume $c_1(TM, J) = (a, b) \in H_2(M, \mathbb{Z})$: if there were an almost complex structure,

$$a^2 + b^2 = c_1^2 = 2\chi + 3\sigma = 14 \tag{5}$$

  which is impossible.

2. Types and Splittings

Let (M, J) be an almost complex structure, J extended $\mathbb{C}$-linearly to $TM \otimes \mathbb{C} = TM^{1,0} \oplus TM^{0,1}$ (with the decomposition being into $+i$ and $-i$ eigenspaces). Here, $TM^{1,0} = \{v - iJv | v \in TM\}$ is the set of *holomorphic* tangent vectors and $TM^{0,1} = \{v + iJv, v \in TM\}$ is the set of *anti-holomorphic* tangent vectors. For instance, on $\mathbb{C}^n$, this gives

$$\frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial z_j},\ \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial \overline{z}_j} \tag{6}$$

respectively. More generally, we have induced real isomorphisms

$$\pi^{1,0} : TM \to TM^{1,0}, v \mapsto v^{1,0} = \frac{1}{2}(v - iJv), \pi^{0,1} : TM \to TM^{0,1}, v \mapsto v^{0,1} = \frac{1}{2}(v + iJv) \tag{7}$$

Then $(Jv)^{1,0} = i(v^{1,0}), (Jv)^{0,1} = -i(v^{0,1})$, so $(TM, J) \cong TM^{1,0} \cong \overline{TM^{0,1}}$ as almost-complex bundles.

Similarly, the complexified cotangent bundle decomposes as $T^*M^{1,0} = \{\eta \in T^*M \otimes \mathbb{C} | \eta(Jv) = i\eta(v)\}, T^*M^{0,1} = \{\eta \in T^*M \otimes \mathbb{C} | \eta(Jv) = -i\eta(v)\}$, with maps from the original cotangent bundle given by

$$\eta \mapsto \eta^{1,0} = \frac{1}{2}(\eta - i(\eta \circ J)) = \frac{1}{2}(\eta + iJ^*\eta), \eta \mapsto \eta^{0,1} = \frac{1}{2}(\eta + i(\eta \circ J)) = \frac{1}{2}(\eta - iJ^*\eta) \tag{8}$$

For $\mathbb{C}^n$, we find that

$$J^*dx_i = dy_i, J^*dy_i = -dx_i \implies dx_j + idy_j = dz_j \in (T^*\mathbb{C}^n)^{1,0}, dx_j - idy_j = d\overline{z_j} \in (T^*\mathbb{C}^n)^{0,1} \tag{9}$$

More generally, on a complex manifold, in holomorphic local coordinates, we have $T^*M^{1,0} = \text{Span}(dz_j)$. Note also that $T^*M^{1,0}$ pairs with $TM^{0,1}$ trivially.

2.1. Differential forms.

Ω^k splits into forms of type (p, q), $p + q = k$, with

$$\wedge^{p,q}T^*M = (\wedge^p T^*M^{1,0}) \otimes (\wedge^q T^*M^{0,1}) = \bigoplus_{p+q=k} \wedge^{p,q}T^*M \tag{10}$$

Definition 1. *For $\alpha \in \Omega^{p,q}(M)$, $\partial\alpha = (d\alpha)^{p+1,q} \in \Omega^{p+1,q}$ and $\overline{\partial}\alpha = (d\alpha)^{p,q+1} \in \Omega^{p,q+1}$.*

In general,

$$d\alpha = (d\alpha)^{p+q+1,0} + (d\alpha)^{p+q,1} + \cdots + (d\alpha)^{0,p+q+1} \tag{11}$$

For a function, we have $df = \partial f + \overline{\partial} f$. Now, say $f : M \to \mathbb{C}$ is J-holomorphic if $\overline{\partial} f = 0 \Leftrightarrow df \in \Omega^{1,0} \Leftrightarrow df(Jv) = idf(v)$.

2.2. Dolbeault cohomology.

Assume d maps $\Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1}$, i.e. $d = \partial + \overline{\partial}$. On $\mathbb{C}^n$, for instance, we have

$$\begin{aligned} \partial(\alpha_{I,J} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}) &= \sum_k \frac{\partial \alpha_{IJ}}{\partial z_k} dz_k \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q} \\ \overline{\partial}(\alpha_{I,J} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}) &= \sum_k \frac{\partial \alpha_{IJ}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q} \end{aligned} \tag{12}$$

Then, $\forall \beta \in \Omega^{p,q}, 0 = d^2\beta = \partial\partial\beta + \partial\overline{\partial}\beta + \overline{\partial}\partial\beta + \overline{\partial}\overline{\partial}\beta \implies \partial^2 = 0, \overline{\partial}^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0$. Since $\overline{\partial}^2 = 0$, we obtain a complex $0 \to \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \cdots$.

Definition 2. *The* Dolbeault cohomology *of M is*

$$H^{p,q}(M) = \frac{\text{Ker } (\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1})}{\text{Im}(\overline{\partial} : \Omega^{p,q-1} \to \Omega^{p,q})} \tag{13}$$

In general, this is not finite-dimensional. We'll see that on a compact Kähler manifold, i.e. a manifold with compatible symplectic and complex structures, $H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$.

2.3. **Integrability.** Let (M, J) be a manifold with almost-complex structure.

Definition 3. *The* Nijenhuis tensor *is the map* $N(u, v) = [Ju, Jv] - J[u, Jv] - J[Ju, v] - [u, v]$ *for* u, v *vector fields on* M*.*

In fact, $N(u, v) = -8\mathrm{Re}([u^{1,0}, v^{1,0}])^{0,1}$.