SYMPLECTIC GEOMETRY, LECTURE 11

Prof. Denis Auroux

1. Chern Classes

Let $E \to M$ be a complex vector bundle, ∇ a connection on E. Recall that we obtain the Chern classes of E via $c(E) = \sum c_i(E) = \det\left(I + \frac{i}{2\pi} R^\nabla\right)$.

Proposition 1. *M compact and oriented $\implies c_r(E) = e(E) \in H^{2r}(M, \mathbb{Z})$.*

Let s be a section transverse to the zero section. Let $Z = s^{-1}(0)$ be its zero set: then

$$[Z] \in H_{2n-2r}(M) \implies PD([Z]) \in H^{2r}(M) \tag{1}$$

is the Euler class of E.

1.1. Chern Classes of Line bundles.

We now restrict to understanding the first Chern class of a line bundle. If M is compact, this is precisely the Euler class. Now, consider a closed, oriented surface Σ: any section vanishes at finitely many points, giving us a well-defined degree by counting these points (with sign). Moreover, we have that $c_1(L) \in H^2(L, \mathbb{Z}) \cong \mathbb{Z}$ is precisely the class s.t. $c_1(L)[\Sigma] = \deg L$. Cut Σ into two parts $U \cup D^2$, where $U = \bigvee S^1$ holds all the non-trivial loops. Any complex bundle over S^1 is trivial, so L is trivial over both U and D^2. To obtain L from $L|_U$ and L_{D^2}, we need to identify $L|_{\partial U} \cong \mathbb{C} \times S^1 \to L|_{\partial D^2} \cong \mathbb{C} \times S^1$. This corresponds to a map $S^1 \to \mathbb{C}^*$ modulo homotopy, i.e. an element of $\pi_1(S^1) \cong \mathbb{Z}$. This is again $\deg L$.

Remark. Alternatively, since L is trivial over D^2 and U, we have a non-vanishing section s of $L|_U$. The Chern class of L measures why this section cannot be extended to all of Σ. Specifically, the Chern class corresponds to the boundary map $\frac{s}{|s|} : \partial D^2 = S^1 \to S^1$.

1.2. Properties of Chern Classes.

Let $c(E) = \sum c_i(E)$ denote the total Chern class of E (with $c_0(E) = 1$).

1. $c(E \oplus F) = c(E) \cup c(F)$.
2. For $f : X \to M$ a smooth map giving a commutative square

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & M \end{array} \tag{2}$$

   where $f^*E = \{(x, v) \in X \times E | f(x) = \pi(v)\}$, we have $c(f^*(E)) = f^*(c(E))$. By the splitting principle, for any $E \to M$, $\exists f : X \to M$ s.t. in the above square, f^* is injective on cohomology, and f^*E splits as a sum of line bundles.

One can define the Chern classes via these properties along with the definition of the first Chern class of a line bundle. Our definition of Chern classes (i.e. via the curvature R^∇) also satisfies these properties.

1. Given bundles E, F with connections ∇^E, ∇^F, the connection on the direct sum is precisely $\nabla^{E\oplus F}(s, t) = (\nabla^E(s), \nabla^F(t))$, implying that the curvature is $R^{E\oplus F} = R^E \oplus R^F$ as desired.
2. Note that, if s is a local section of E near $f(x)$, then $s \circ f$ is a local section of f^*E near x. By the definition of the pullback connection, $\nabla^{f^*E}(f^*(s)) = f^*(\nabla^E s)$. Via the definition of curvature, we see that $f^*(R^\nabla) = R^{f^*\nabla}$ as well, implying the desired pullback property.

Remark. $c_1(L) \in H^2(M, \mathbb{Z})$ completely classifies $\mathbb{C}$-line bundles. Moreover, it defines a group isomorphism between the set of line bundles over M under $\otimes$ with $H^2(M, \mathbb{Z})$. To see this, recall that a line bundle is precisely a collection of local trivializations $\{f_\alpha : L|_{U_\alpha} \stackrel{\cong}{\to} U_\alpha \times \mathbb{C}\}$ with attaching maps $g_{\alpha,\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathbb{C}^*)$ satisfying the cocycle condition

$$g_{\alpha,\beta} g_{\beta,\gamma} g_{\gamma\alpha} = 1 \tag{3}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$. This corresponds precisely with the Cech cohomology on M, where $\{g_{\alpha,\beta}\}$ is a 1-cocycle. In this description, c_1 is the connecting map in the long exact sequence

$$\cdots \to 0 = H^1(M, \underline{\mathbb{C}}) \to H^1(M, \underline{\mathbb{C}}^*) \to^{c_1} H^2(M, \mathbb{Z}) \to H^2(M, \underline{\mathbb{C}}) = 0 \to \cdots \tag{4}$$

associated to the short exact sequence of sheaves $0 \to \mathbb{Z} \to \underline{\mathbb{C}} \stackrel{\exp}{\to} \underline{\mathbb{C}}^* \to 0$ where $\underline{\mathbb{C}}, \underline{\mathbb{C}}^*$ are the sheaves of $\mathbb{C}^\infty$ functions with values in $\mathbb{C}, \mathbb{C}^*$. One can also see directly the fact that $c_1(L \otimes L') = c_1(L) + c_1(L')$ using the definition of the tensor product connection $\nabla^{L \otimes L'} = \nabla^L \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^{L'}$.

Now, for (M, ω) a symplectic manifold, J a compatible almost-complex structure, (TM, J) is a complex vector bundle, with $c_j(TM) \in H^{2j}(M, \mathbb{Z})$. Since the RHS is discrete, we get an invariant of the almost-complex structure up to deformation, and since the space of compatible J's is connected, the complex isomorphism class of (TM, J) is uniquely determined. Explicitly, if J_t is a family of complex structures on E, the map $\phi : v \mapsto \frac{1}{2}(v - J_t J_{t_0} v)$ is a complex isomorphism from (E, J_{t_0}) to (E, J_t) since

$$\phi(J_{t_0} v) = \frac{1}{2}(J_{t_0} v + J_t v) = J_t(\frac{1}{2}(v - J_t J_{t_0} v)) = J_t \phi(v) \tag{5}$$

Thus, $c_j(TM, J)$ is independent of the choice of almost-complex structure (it is even an invariant of the deformation class of M): for instance, $c_n(TM) \in H^{2n}(M, \mathbb{Z}) \cong \mathbb{Z}$ is an invariant of the manifold (the *Euler characteristic*).

Remark. For $1 \leq j \leq n-1$, c_j does depend on the choice of symplectic structure, however: there exists a 4-manifold M with symplectic forms ω_1, ω_2 s.t. $c_1(TM, \omega_1) \neq c_1(TM, \omega_2)$.