SYMPLECTIC GEOMETRY, LECTURE 9

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1. Curvature

Let ∇ be a connection as before: then we get a curvature tensor $R^\nabla \in \Omega^2(M, \mathrm{End}(E))$, i.e. a matrix of 2-forms (in local coordinates).

Definition 1. *Given a local section σ and vector fields U, V,*

$$R^\nabla(U,V)\sigma = \nabla_U \nabla_V \sigma - \nabla_V \nabla_U \sigma - \nabla_{[U,V]}\sigma \tag{1}$$

Proposition 1. *R^∇ is a tensor (i.e. it is defined pointwise, depending only on $\sigma(x)$ and not on its derivatives).*

Remark. Use local coordinates, let $f_i = \frac{\partial}{\partial x_i}$: then $R^\nabla = \sum_{i<j}(\nabla_{f_i}\nabla_{f_j}\sigma - \nabla_{f_j}\nabla_{f_i}\sigma)dx_i \wedge dx_j$.

In a local trivialization (e_i), $\nabla = d + A$, $A \in \Omega^1(\mathrm{End}\ E)$, i.e. $\nabla e_j = \sum_i a_{ij} e_i$. Then

$$\begin{aligned} R^\nabla(U,V)e_j &= \nabla_U(\sum_i a_{ij}(V)e_i) - \nabla_V(\sum_i a_{ij}(U)e_i) - \sum_i a_{ij}([U,V])e_i \\ &= \sum_i (U\cdot a_{ij}(V))e_i + \sum_{i,k} a_{ki}(U)a_{ij}(V)e_k - \sum_i (V \cdot a_{ij}(U))e_i - \sum_{i,k} a_{ki}(V)a_{ij}(U)e_k - \sum_i a_{ij}([U,V])e_i \end{aligned} \tag{2}$$

The component along e_i is

$$R^\nabla_{ij}(U,V) = U\cdot a_{ij}(V) - V\cdot a_{ij}(U) - a_{ij}([U,V]) + \sum_\ell a_{i\ell}(U)a_{\ell j}(V) - a_{i\ell}(V)a_{\ell j}(U) \tag{3}$$

That is, $R^\nabla_{ij} = da_{ij} + \sum_\ell a_{i\ell} \wedge a_{\ell j}$.

Remark. We can take this as the definition of R^∇, i.e. write $R^\nabla = dA + A \wedge A$.

If we change trivializations $(e_1, \dots, e_n) \mapsto (e'_1, \dots e'_n)$ via $e'_j = \sum g_{ij} e_i$ s.t. g is a matrix-valued function, then we can write $s = \sum \xi'_j e'_j = \sum g_{ij}\xi'_j e_i$. In other words, if s corresponds to a vector ξ in the trivialization (e_i) and ξ' in the trivialization e'_i, then $\xi = g\xi'$, i.e. $\xi_i = \sum g_{ij}\xi'_j$. Now we write ∇s in the trivialization: $\nabla(\xi)_e = (d\xi + A\xi)_e$ for A the connection 1-form in the trivialization (e_i). Changing trivializations gives $\nabla(g^{-1}\xi)_{e'} = g^{-1}(d\xi + A\xi)|_{e'}$, (???)

i.e. $\nabla(\xi')_{e'} = (g^{-1}(d(g\xi') + Ag\xi'))_{e'} = d\xi' + (g^{-1}Ag + g^{-1}d(g))\xi'$. Thus, the connection 1-form in the new trivialization is $A' = g^{-1}Ag + g^{-1}dg$ (as matrix-valued form).

Proposition 2. *$dA' + A' \wedge A' = g^{-1}(dA + A\wedge A)g$, i.e. $R^\nabla = dA + A \wedge A$ is a well-defined element of $\Omega^2(M, \mathrm{End}E)$ independently of trivialization.*

Recall that the product here is matrix multiplication, with the entries 1-forms multiplied under $\wedge$.

Proof. First, $gA' = Ag + dg$: taking exterior derivatives, we get $dg \wedge A' + gdA' = dA\cdot g - A\wedge dg + 0$. Thus,

$$\begin{aligned} g(dA' + A'\wedge A') &= (dA\cdot g - A\wedge dg - dg \wedge A') + gA'\wedge A' \\ &= dA\cdot g - A\wedge dg - dg\wedge A' + (Ag + dg)\wedge A' \\ &= dA\cdot g - A \wedge dg + Ag \wedge A' \\ &= dA \cdot g - A\wedge dg + A \wedge (dg + Ag) = (dA + A\wedge A)g \end{aligned} \tag{4}$$

□