SYMPLECTIC GEOMETRY, LECTURE 6

Prof. Denis Auroux

1. Applications

(1) The work done last time gives us a new way to look at $T_{\mathrm{id}}\mathrm{Symp}(M,\omega)$ (using C^1-topology, wherein $f_i : X \to Y$ converges to f iff $f_i \to f$ uniformly on compact sets and same for $df_i : TX \to TY$. Now, $f \in \mathrm{Symp}(M,\omega)$ gives a graph $\mathrm{graph}(f) = \{(x, f(x))\} \subset (M \times M, \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega)$ which is a Lagrangian submanifold. If f is C^1-close to the identity map, then $\mathrm{graph}(f)$ is C^1-close to the diagonal $\Delta = \{(x,x)\} \subset (M \times M, \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega)$ (i.e. the graph of the identity map). By Weinstein, a tubular neighborhood of Δ is diffeomorphic to $U_0 \subset (T^*M, \omega_{T^*M})$, and the graph of f gives a section (C^1-close to the zero section), i.e. the graph of a C^1-small $\mu \in \Omega^1(M)$. The fact that its graph is Lagrangian implies that μ is closed, i.e. $d\mu = 0$. Thus, we have an identification $T_{\mathrm{id}}(\mathrm{Symp}(M,\omega)) \cong \{\mu \in \Omega^1 | d\mu = 0\}$ with C^1 topologies.

(2)

Theorem 1. *For (M,ω) compact, if $H^1(M,\mathbb{R}) = 0$, then every symplectomorphism of M which is C^1 close to the identity has ≥ 2 fixed points.*

Theorem 2. *For (M,ω) symplectic, $X \subset (M,\omega)$ compact and Lagrangian, if $H^1(X,\mathbb{R}) = 0$, then every Lagrangian submanifold of M which is C^1 close to X intersects X in ≥ 2 points.*

The first theorem follows from the second, using the diagonal embedding $\Delta \subset M \times M$. To see the second theorem, note that $H^1(X) = 0$ implies that, given any graph $Y = \mathrm{graph}(\mu)$ C^1-close to X with $d\mu = 0$, we have $\mu = dh$ for some $h : X \to \mathbb{R}$. Since such an h must have at least 2 critical points, $\exists$ at least 2 points at which $\mu = 0$, i.e. points at which Y intersects X.

2. Arnold Conjecture

Arnold's conjecture: Let (M,ω) be compact, $f \in \mathrm{Ham}(M,\omega)$ the time 1 flow of X_{H_t} for $H_t : M \to \mathbb{R}$ a 1-periodic Hamiltonian ($H : M \times \mathbb{R} \to \mathbb{R}$ smooth with $H_{t+1} = H_t$). Then the number of fixed points of f is at least the minimal number of critical points of a smooth function on M. Moreover, assume the fixed points of f are nondegenerate, i.e. if $f(x) = x$ then $\det\,(d_x f - \mathrm{id}) \neq 0$. Then $\#\mathrm{Fix}(f)$ is at least the minimal number of critical points of a Morse function on M, which in turn is $\geq \sum_i \dim\, H^i(M)$.

Remark. The last inequality follows from classical Morse theory. Given a Morse function f on a manifold M (equipped with a Riemannian metric satisfying the Morse-Smale condition), we have the Morse complex C^i generated by critical points of index i, and the Morse differential $d : C^i \to C^{i+1}$ which counts gradient trajectories between critical points. Then $H^*(C^*, d) \simeq H^*(M)$, so $\#\mathrm{Fix}(f) = \sum \dim\, C^i \geq \sum \dim\, H^i$.

The case where $H_t = H$ is independent of t is easy: if p is a critical point of H then $X_H(p) = 0$ so the flow f fixes p. The general case was proved by Conley-Zehnder, Floer, Hofer-Salamon, Ono, Fukaya-Ono, Li-Tian, ... using *Floer homology.* Floer homology is formally the ∞-dimensonal Morse theory of a functional on a covering of the loop space, $\widetilde{\Omega M} = \{\gamma : S^1 \to M$ contractible + homotopy class of disc with $\partial D = \gamma\}$:

$$\mathcal{A}_H : \widetilde{\Omega M} \to \mathbb{R}, \quad \mathcal{A}_H(\gamma) = -\int_{D^2} u^*\omega - \int_{S^1} H(t, \gamma(t))\, dt \tag{1}$$

where the first term involves $u : D^2 \to M$ with $u(\partial D) = \gamma$ in the given homotopy class.

Given $v : S^1 \to \gamma^*TM$ (a vector field along γ), the differential of $\mathcal{A}_H$ is given by

$$D\mathcal{A}_{H\,(\gamma)}(v) = -\int_{S^1} \omega(v(t), \dot\gamma(t))\,dt - \int_{S^1} dH_{t\,(\gamma(t))}(v(t))\,dt = \int_{S^1} (i_{\dot\gamma(t)}\omega - dH_t)(v(t))\,dt.$$

Since $dH_t = i_{X_t}\omega$, this vanishes $\forall v$ if and only if $\dot\gamma(t) = X_t(\gamma(t))$, i.e. γ is a periodic orbit of the flow. Hence critical points of $\mathcal{A}_H$ correspond to fixed points of f. Moreover, formally gradient trajectories of $\mathcal{A}_H$ correspond to solutions $u : \mathbb{R} \times S^1 \times M$, $(s,t) \mapsto u(s,t)$ of the PDE

$$\frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - \nabla H_t(u)\right) = 0. \tag{2}$$