SYMPLECTIC GEOMETRY, LECTURE 5

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Last time we proved:

Theorem 1 (Moser). *Let M be a compact manifold, (ω_t) symplectic forms, $[\omega_t]$ constant $\implies$ $(M, \omega_0) \cong (M, \omega_1)$.*

Theorem 2 (Darboux). *Locally, any symplectic manifold is locally isomorphic to $(\mathbb{R}^{2n}, \omega_0)$.*

1. Tubular Neighborhoods

Let $M^n \supset X^k$ be a submanifold with inclusion map i. Then we get a map $d_x i : T_x X \hookrightarrow T_x M$, with associated *normal space* $N_x X = T_x M / T_x X$. Note that if there is a metric, one can identify this with the orthogonal space to X at x. Putting all these spaces together, we get a *normal bundle* $NX = \{(x, v) | x \in X, v \in N_x X\}$ with zero section $i_0 : X \to NX, x \mapsto (x, 0)$.

Theorem 3. *$\exists U_0$ a neighborhood of X in NX (via the 0-section) and U_1 a neighborhood of X in M s.t. $\exists \phi : U_0 \xrightarrow{\sim} U_1$ a diffeomorphism.*

Proof. (Idea) Equip M with a Riemannian metric g, so $N_x X \xrightarrow{\sim} T_x X^\perp \subset T_x M$. Then, given $x \in X, v \in N_x X$ for $|v|$ sufficiently small ($|v| = \sqrt{g(v,v)} < \epsilon$), we obtain an exponential function $\exp_x(v)$ (defined by considering a small geodesic segment with origin x and tangent vector v). We obtain a map $U_0 \to M, (x, v) \mapsto \exp_x(v)$. For $x \in X, T_{(x,0)}(NX) = T_x X \oplus N_x X$ and

$$d_{(x,0)} \exp(u, v) = u + v \in T_x X \oplus T_x X^\perp \tag{1}$$

this giving us a local diffeomorphism near the 0-section. Thus, locally on some neighborhood of the 0-section in NX, exp induces a diffeomorphism onto $\exp(U_0)$ = neighborhood of X in M. □

Let $U_1 = \{\exp_x(v) | \, |v| < \epsilon'(x)\} \subset M$ be a tubular neighborhood of X in M as constructed above, with $U_0 \subset NX$ the corresponding neighborhood of the zero section. Via the projection $\pi : U_0 \to X$, whose fibers are balls in $\mathbb{R}^{n-k}$, we see that U_1 retracts onto X, i.e. we have a null-homotopic map $U_1 \xrightarrow{\pi} X \xrightarrow{i} U_1$.

Corollary 1. *$i^* : H^*(U_1, \mathbb{R}) \to H^*(X, \mathbb{R})$ is an isomorphism.*

Proposition 1. *$\beta \in \Omega^\ell(U), d\beta = 0, i^*\beta = \beta|_X = 0 \implies \exists \mu \in \Omega^{\ell-1}(U), \beta = d\mu$ and $\mu_x = 0 \ \forall x \in X$.*

Proof. Identify $U \cong U_0 \subset NX$, set $\rho_t : (x, v) \mapsto (x, tv)$, and let

$$\mu_{(x,v)} = \int_0^1 \rho_t^*(i_{(0,v)}\beta) dt \tag{2}$$

Then $\mu = 0$ on the zero section, and

$$d\mu = \int_0^1 \rho_t^*(d i_{X_t} \beta) dt \tag{3}$$

where $X_t(x, tv) = (0, v)$. Since β is closed, $d i_{X_t}\beta = L_{X_t}\beta$, so

$$d\mu = \int_0^1 \frac{d}{dt}(\rho_t^*\beta) dt = \rho_1^*\beta - \rho_0^*\beta = \beta - \pi^* i^* \beta = \beta \tag{4}$$

□

Theorem 4 (Local Moser). *Let $X \hookrightarrow M$ be a submanifold, ω_0, ω_1 symplectic forms on M s.t. $(\omega_0)_p = (\omega_1)_p \forall p \in X$. Then $\exists$ neighborhoods $U_0, U_1 \supset X$ and $\phi : U_0 \xrightarrow{\sim} U_1$ s.t. $\phi^*\omega_1 = \omega_0$ and $\phi|_X = \mathrm{id}$.*

That is, we have a symplectomorphism $(U_0, \omega_0) \xrightarrow{\sim} (U_1, \omega_1)$ commuting with the inclusion of X.

Proof. Let U_0 be a tubular neighborhood of X. Since $\omega_1 - \omega_0$ is closed and is 0 on X, by the above proposition we have a form $\mu \in \Omega^1(U_0)$ s.t. $\omega_1 - \omega_0 = d\mu$ and μ is 0 along X. Now, let $\omega_t = (1-t)\omega_0 + t\omega_1$: these form a family of closed two-forms which are ω_0 along X and thus nondegenerate at X. Since nondegeneracy is an open condition, $\exists U_0' \subset U_0$ on which ω_t is symplectic $\forall t$. $\exists v_t$ a vector field on U_0' s.t. $i_{v_t}\omega_t = -\mu$ with $v_t = 0$ along X. Letting ρ_t be the flow of v_t, we find that ρ_t is the identity along X, and $\exists$ a neighborhood U_0'' on which the flow is well defined. Finally,

$$\frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*\left(L_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) = \rho_t^*(-d\mu + (\omega_1 - \omega_0)) = 0 \tag{5}$$

□

Proposition 2. *Let $X \hookrightarrow (M, \omega)$ be a Lagrangian submanifold. Then $NX \xrightarrow{\sim} T^*X$.*

Proof. $E \subset (V, \Omega)$ a Lagrangian subspace $\implies \Omega : V \xrightarrow{\sim} V^* \twoheadrightarrow E^*, v \mapsto \Omega(v, \cdot)$ is onto with kernel $\cong E^{\perp\Omega} = E$, so $V/E \cong E^*$. □

Theorem 5 (Weinstein's Lagrangian Neighborhood)**.** *Let (M, ω) be a symplectic manifold, $i : X \hookrightarrow M$ a closed Lagrangian submanifold, $i_0 : X \to (T^*X, \omega_0)$ the zero-section. Then $\exists U_0$ a neighborhood of X in T^*X and U a neighborhood of X in M s.t. we have a symplectomorphism $(U_0, \omega_0) \xrightarrow{\sim} (U, \omega)$ which is the identity on X.*

Proof. $NX \cong T^*X$, so $\exists N_0 \supset X$ in $T^*X, N \supset X$ in M, and a diffeomorphism $\psi : N_0 \xrightarrow{\sim} N$ which preserves X. Now, let ω_0 be the canonical form on T^*X and $\omega_1 = \psi^*\omega$. These are both sympectic forms on $N_0 \subset T^*X$ s.t. the zero section X is Lagrangian for both.

We claim that we can build (canonically) a family of isomorphisms $L_p : T_pN_0 \to T_pN_0$ for $p \in X$ s.t. $L_{p|T_pX} = \mathrm{id}$ and $(L_p^*\omega_1)_p = (\omega_0)_p$. By Whitney's extension theorem, $\exists$ a neighborhood $N' \supset X$ and an embedding $h : N' \hookrightarrow N_0$ s.t.

$$h|_X = \mathrm{id}, dh_p = L_p \forall p \in X \tag{6}$$

(Idea: use a Riemannian metric, and set $h(p, \xi) = \exp_{p,0} L_p(0, \xi)$). Then $\forall p \in X, (h^*\omega_1)_p = (\omega_0)_p$, so we can use local Moser for $h^*\omega_1$ and ω_0. We therefore obtain $U_0, U_1 \supset X$ and a local symplectomorphism $f : (U_0, \omega_0) \xrightarrow{\sim} (U_1, h^*\omega_1)$. Setting $\phi = \psi \circ h \circ f$ gives us the desired result.

To prove the claim, decompose $T_{(p,0)}N_0 = T_pX \oplus T_p^*X$, with a chosen basis for T_pX and the dual basis for T_p^*X. We have two symplectic forms on this space, namely $\omega_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \omega = \begin{pmatrix} 0 & -B^t \\ B & C \end{pmatrix}$. That is, we know that

$$\omega_0((v_1, \xi_1), (v_2, \xi_2)) = \xi_1(v_2) - \xi_2(v_1) \tag{7}$$

and $\omega|_{T_pX} = 0$. We want to find a matrix $L = \begin{pmatrix} I & * \\ 0 & * \end{pmatrix}$ s.t. $L^t\omega L = \omega_0$. Setting

$$L = \begin{pmatrix} I & -\frac{1}{2}B^{-1}CB^{-t} \\ 0 & B^{-t} \end{pmatrix} \tag{8}$$

gives the desired matrix: furthermore, the construction doesn't depend on the choice of basis. □