SYMPLECTIC GEOMETRY, LECTURE 2

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1. Homology and Cohomology

Recall from last time that, for M a smooth manifold, we produced a graded differential algebra $(\Omega^*(M), \wedge, d)$ giving us a cohomology $H^*(M)$ with cup product $[\alpha] \cup [\beta] = [\alpha \wedge \beta]$ (which is well-defined since $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$ and $(\alpha + d\eta) \wedge \beta = \alpha \wedge \beta + d\eta \wedge \beta$). Furthermore, we obtain a pairing with homology: for $\Sigma \subset M$ a p-dimensional, oriented, closed submanifold with associated class $[\Sigma] \in H_p(M)$, we define

$$\langle [\alpha], [\Sigma] \rangle = \int_\Sigma \alpha \tag{1}$$

for $[\alpha] \in H^p(M, \mathbb{R})$, and extend this by linearity to give a pairing with all of $H_p(M)$. That this is well-defined is a consequence of Stokes' theorem:

$$\int_\Sigma d\alpha = \int_{\partial \Sigma} \alpha \tag{2}$$

Remark. A form is closed $\Leftrightarrow$ its integral on submanifolds depends only the homology class of the submanifold.

Furthermore, if M^n is compact, closed, and oriented, we have a nondegenerate pairing

$$H^p(M, \mathbb{R}) \otimes H^{n-p}(M, \mathbb{R}) \to \mathbb{R}, [\alpha] \otimes [\beta] \mapsto \int_M \alpha \wedge \beta \tag{3}$$

which induces the Poincaré duality $H^{n-p}(M, \mathbb{R}) \to H_p(M, \mathbb{R})$. In the noncompact case, we have the same statement using cohomology with compact support $H_C^{n-p}(M, \mathbb{R})$.

2. Symplectic Vector Spaces

Let V be a f.d. vector space $/\mathbb{R}$.

Definition 1. *A* symplectic structure *on V is a bilinear, non-degenerate, skew-symmetric pairing $\Omega : V \times V \to \mathbb{R}$. That is, as a matrix, it is invertible and skew-symmetric.*

Example. For $\mathbb{R}^{2n}$ with basis $\{e_i\}_{i=1}^n, \{f_i\}_{i=1}^n$, we have a standard symplectic form given by $\Omega_0(e_i, e_j) = \Omega_0(f_i, f_j) = 0 \ \forall i, j, \Omega_0(e_i, f_j) = \delta_{i,j} = -\Omega_0(f_j, e_i)$. As a matrix, it is given by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Definition 2. *For $E \subset V$ a linear subspace, Ω a bilinear form, the* orthogonal complement of E *is $E^\Omega = E^\perp = \{v \in V | \Omega(u, v) = 0 \ \forall u \in E\}$.*

Note that Ω is non-degenerate $\Leftrightarrow V^\Omega = \{0\}$.

Example. In $\mathbb{R}^{2n}$ with basis as above,

$$\begin{aligned} \mathrm{Span}\{e_1\}^\Omega &= \mathrm{Span}\{e_1, \ldots, e_n, f_2, \ldots, f_n\} \\ \mathrm{Span}\{e_1, f_1\}^\Omega &= \mathrm{Span}\{e_2, \ldots, e_n, f_2, \ldots, f_n\} \\ \mathrm{Span}\{e_1, \ldots, e_n\}^\Omega &= \mathrm{Span}\{e_1, \ldots, e_n\} \end{aligned} \tag{4}$$

Definition 3. *A* standard (symplectic) basis *of (V^{2n}, Ω) is a basis $(\{e_i\}, \{f_i\})$ satisfying the above.*

Theorem 1. *For (V^n, Ω) a symplectic vector space, $\exists$ a standard basis.*

Proof. We induce on n: the base case is trivial. Choose some vector $e_1 \in V \setminus \{0\}$. By nondegeneracy, $\Omega(e_i, \cdot) \neq 0 \implies \exists f_1$ s.t. $\Omega(e_1, f_1) = 1$. Let $W = \text{Span}\{e_1, f_1\}^\Omega$: then $\Omega|_W$ is symplectic since $u \in W, \Omega(u, q) = 0\ \forall w \in W \implies \Omega(u, e_1) = 0, \Omega(u, f_1) = 0 \implies u = 0$. Furthermore, $V = \text{Span}\{e_1, f_1\} \oplus W$. To see this, note first that, if $v = ae_1 + bf_1 \in W, \Omega(e_1, v) = b = 0$ and $\Omega(f_1, v) = a = 0$, so $W \cap \text{Span}\{e_1, f_1\} = \varnothing$. Secondly, for $v \in V$, we can write $v = w + ae_1 + bf_1$, where $w = v - \Omega(e_1, v)f_1 + \Omega(f_1, v)e_1 \in W$. Since W has dimension $n - 2$, we are done. ☐

Corollary 1. *V symplectic $\implies$ V is even-dimensional and* symplectomorphic *to* $(\mathbb{R}^{2n}, \Omega_0)$.

We denote the symplectic automorphisms of (V, Ω) by $\text{Sp}(V, \Omega) = \text{Sp}(2n, \mathbb{R})$.

Remark. dim E^Ω = dim V − dim E because $V \xrightarrow{\cong} V^* \to E^*, v \mapsto \Omega(v, \cdot) \mapsto \Omega(v, \cdot)|_E$ is surjective with kernel E^Ω.

Definition 4. *$E \subset V$ is a* symplectic subspace *if $\Omega|_E$ is nondegenerate, e.g. in a standard basis E is the span of*

$$(e_1, f_1, \ldots, e_k, f_k) \tag{5}$$

Problem. Prove that E is a symplectic subspace $\Leftrightarrow E \cap E^\Omega = \{0\} \Leftrightarrow V = E \oplus E^\Omega$.

Definition 5. *$E \subset V$ is an* isotopic (resp. coisotopic, lagrangian) subspace *if $E \subset E^\Omega$ (resp. $E^\Omega \subset E, E^\Omega = E$), e.g. in a standard basis E is the span of $(e_1, \ldots, e_k)$ (resp. $(e_1, f_1, \ldots, e_k, f_k, e_{k+1}, \ldots e_n)$, $(e_1, \ldots, e_n)$).*

Example. For $E \subset V$ Lagrangian with basis $(e_1, \ldots, e_n)$, we can complete this to a symplectic basis

$$(e_1, \ldots, e_n, f_1, \ldots, f_n) \tag{6}$$

of V.

Definition 6. *The* symplectic volume form *is $\frac{1}{n!}\Omega^{\wedge n}$ (where Ω is considered as an element of $\bigwedge^2(V^*)$.*

Note that, since Ω is nondegenerate, we can write $\Omega = \sum_i e^i \wedge f^i$, so $\Omega^{\wedge n} = n! e^1 \wedge f^1 \wedge \cdots \wedge e^n \wedge f^n$ is a non-zero top form, and our volume form is well-defined. In fact, $\Omega^{\wedge n} \neq 0 \Leftrightarrow \Omega$ is nondegenerate.

3. Symplectic Manifolds

Let M be a smooth manifold.

Definition 7. *A* symplectic form *on M is a 2-form ω (i.e. a skew-symmetric pairing $\omega_p : T_pM \times T_pM \to \mathbb{R}$ for all $p \in M$) which is nondegenerate (i.e. $\frac{1}{n!}\omega^n$ is a volume form) and closed (i.e. $d\omega = 0$).*

Remark. M symplectic $\implies$ it is even-dimensional and naturally oriented. Moreover, $[\omega] \in H^2(M, \mathbb{R})$ plays an important role, especially if M is compact, as in this case $\int_M \frac{\omega^n}{n!} = \text{vol}(M) > 0 \implies [\omega] \neq 0$.

Example. For $\mathbb{R}^{2n}, \omega_0 = \sum dx_i \wedge dy_i$ is the standard symplectic structure: for $\mathbb{C}^n$, we write this as $\omega = \frac{i}{2} \sum dz_j \wedge d\overline{z_j}$ instead. Furthermore, for an orientable surface Σ, any area form is a symplectic form.

Problem. For which values of n does S^{2n} (resp. T^{2n}) have a symplectic structure?

Definition 8. *A* symplectomorphism *is a diffeomorphism $\phi : (M, \omega) \to (M', \omega')$ s.t. $\phi^*\omega' = \omega$.*

We denote the group of symplectomorphisms of M by $\text{Symp}(M, \omega)$.

Example. For $S^2 \subset \mathbb{R}^3$, $\text{Symp}(S^2)$ is the group of area and orientation preserving diffeomorphisms, which is much larger than the group of isometries.

Theorem 2 (Darboux)**.** *Every symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega)$, i.e. it has local coordinates in which $\omega = \sum dx_i \wedge dy_i$.*