18.966 – Homework 2 – Solutions.

1. Equip $\mathbb{R}^7 = \operatorname{Im}\mathbb{O} = \{a + be,\ \operatorname{Re} a = 0\}$ with the cross-product $x \times y = \operatorname{Im}(xy)$. By definition of the octonion product, if $x, y \in \operatorname{Im}\mathbb{O}$ then $\operatorname{Re}(xy) = -\langle x, y\rangle$ (the usual Euclidean scalar product on $\mathbb{R}^7$). Indeed,

$$\operatorname{Re}((a + be)(a' + b'e)) = \operatorname{Re}(aa' - \overline{b'}b) = \operatorname{Re}(-a\overline{a'} - \overline{b'}b) = -\langle a, a'\rangle - \langle b, b'\rangle.$$

Therefore $\|x \times y\| = \|\operatorname{Im}(xy)\| \le \|xy\| = \|x\|\|y\|$, with equality iff $x \perp y$. Let $x \in S^6 \subset \mathbb{R}^7$, and let $y \in T_xS^6 \simeq x^\perp \subset \mathbb{R}^7$. Then we define

$$J_x(y) = x \times y.$$

Note that, since $y \perp x$, we have $J_x(y) = x \times y = xy + \langle x, y\rangle = xy$.

We have to prove that J_x maps T_xS^6 to itself, and that $J_x^2 = -\mathrm{Id}$. For this purpose, let $x = a + be$ be a unit imaginary octonion ($\bar{a} = -a$) and let $y = c + de$ be any octonion: then

$$\begin{aligned} x(xy) &= a(ac - \bar{d}b) - (\bar{a}\bar{d} + c\bar{b})b + (da + b\bar{c})ae + b(\bar{c}\bar{a} - \bar{b}d)e \\ &= -|a|^2c - (a + \bar{a})\bar{d}b - c|b|^2 - d|a|^2e + b\bar{c}(a + \bar{a})e - |b|^2de \\ &= -(|a|^2 + |b|^2)(c + de) = -\|x\|^2y = -y. \end{aligned}$$

If $y \in T_xS^6$, i.e. y is imaginary and $y \perp x$, then $\langle x, xy\rangle = -\operatorname{Re}(x(xy)) = \operatorname{Re}(y) = 0$, so J_x maps T_xS^6 to itself, and $J_x^2 = -\mathrm{Id}$.

2. a) Any vector in JL is of the form Ju, with $u \in L$. Given $u, v \in L$, we have $\Omega(Ju, Jv) = \Omega(u, v) = 0$ (since J is Ω-compatible and L is Lagrangian), and $\dim JL = \dim L = \frac{1}{2}\dim V$, so JL is Lagrangian. Also, $\forall u, v \in L$, $g(u, Jv) = \Omega(u, J(Jv)) = -\Omega(u, v) = 0$, so any vector in L is orthogonal to any vector in JL, i.e. $JL \subset L^\perp$. Since $\dim JL = \frac{1}{2}\dim V = \dim L^\perp$, we conclude that $JL = L^\perp$.

b) Assume J is Ω-compatible, and let L be a Lagrangian subspace of (V, Ω). Choose a g-orthonormal basis $(e_1, \dots, e_n)$ of L, and let $f_i = Je_i \in JL$. Then $\Omega(e_i, e_j) = 0$ since L is Lagrangian, and $\Omega(f_i, f_j) = 0$ since JL is Lagrangian. Moreover, $\Omega(e_i, f_j) = \Omega(e_i, Je_j) = g(e_i, e_j) = \delta_{ij}$. Hence we have a standard basis with $f_i = Je_i$.

Conversely, if there exists a standard basis with $f_i = Je_i$, then $\Omega(e_i, Je_j) = \Omega(f_i, Jf_j) = \delta_{ij}$, and $\Omega(e_i, Jf_j) = \Omega(f_i, Je_j) = 0$, so the bilinear form $g = \Omega(\cdot, J\cdot)$ is symmetric and definite positive (and $(e_1, \dots, e_n, f_1, \dots, f_n)$ is an orthonormal basis). Hence J is Ω-compatible.

3. a) Recall that, for any vector $u \in T_xM$, $\nabla s(u)$ is the vertical component of $ds_x(u) \in T_{s(x)}L$ (while the horizontal component of $ds_x(u)$ is the horizontal lift of u). Therefore the assumption that ∇s is surjective at every point of $Z = s^{-1}(0)$ means that the graph Γ_s of s is transverse to the zero section $\Gamma_0 \subset L$, and hence that $Z = \Gamma_s \cap \Gamma_0$ is smooth. Moreover, at every point x of Z we have $T_xZ = T_x\Gamma_s \cap T_x\Gamma_0$, i.e. the tangent space to Z is the set of all vectors $v \in T_xM$ such that $ds_x(v)$ is tangent to the zero section, i.e. horizontal, i.e. $\nabla s(v) = 0$. Hence $TZ = \operatorname{Ker} \nabla s$.

b) Let $x \in Z$, and assume that $|\partial s_x| > |\bar\partial s_x|$. We want to show that the restriction of $T_xZ = \operatorname{Ker} \nabla s_x$ is a symplectic subspace of (T_xM, ω_x). This is a linear algebra question involving the linear map $\nabla s_x : T_xM \to L_x$.

Use a unit length element in L_x to identify the fiber L_x (a rank 1 complex vector space with a Hermitian norm) with $\mathbb{C}$ equipped with the standard norm $|\cdot|$. Then ∇s_x becomes a linear map $T_xM \to \mathbb{C}$.

Method 1: Let $g : T_xM \times T_xM \to \mathbb{R}$ be the metric induced by ω and J, and consider the linear form $\partial s_x : T_xM \to \mathbb{C}$. There exists a unique vector $u \in T_xM$ such that $\operatorname{Re} \partial s_x = g(u, \cdot)$; because $\partial s_x \circ J = i\partial s_x$, we have $\operatorname{Im} \partial s_x = g(-Ju, \cdot)$. Similarly, there exists a unique $v \in T_xM$ such that $\operatorname{Re} \bar\partial s_x = g(v, \cdot)$, and $\operatorname{Im} \bar\partial s_x = g(Jv, \cdot)$. The assumption $|\partial s_x| > |\bar\partial s_x|$ is equivalent to the property $g(u, u) > g(v, v)$.

Since $\nabla s_x = \partial s_x + \bar\partial s_x$, we have $\operatorname{Re} \nabla s_x = g(u + v, \cdot) = \omega(-Ju - Jv, \cdot)$, and $\operatorname{Im} \nabla s_x = g(-Ju + Jv, \cdot) = \omega(-u + v, \cdot)$. Hence, $E = T_xZ = \operatorname{Ker} \nabla s_x$ is the set of all tangent vectors that are symplectically orthogonal to $-Ju-Jv$ and $-u+v$, i.e. $E^\omega = \operatorname{span}(-Ju-Jv, -u+v)$. Recall that $E \subset (T_xM, \omega)$ is a symplectic subspace $\Leftrightarrow T_xM = E \oplus E^\omega \Leftrightarrow E^\omega$ is a symplectic subspace. So we just need to check that the restriction of ω to E^ω is non-degenerate. Since

$$\begin{aligned}\omega(-u + v, -Ju - Jv) &= \omega(u, Ju) - \omega(v, Ju) + \omega(u, Jv) - \omega(v, Jv) \\ &= g(u, u) - g(v, u) + g(u, v) - g(v, v) = g(u, u) - g(v, v) > 0,\end{aligned}$$

we conclude that Z is a symplectic submanifold of (M, ω).

Method 2: use the result of Problem 2 to identify (T_xM, ω, J, g) with $(\mathbb{C}^n, \omega_0, i, |\cdot|)$. Then $\partial s_x : \mathbb{C}^n \to \mathbb{C}$ can be written as $\partial s_x(u_1, \dots, u_n) = \sum \alpha_j u_j$ for some constants $\alpha_i \in \mathbb{C}$, and similarly $\bar\partial s_x(u_1, \dots, u_n) = \sum \beta_j \bar u_j$. In order to prove that T_xZ is a symplectic subspace, we consider a non-zero vector $u = (u_1, \dots, u_n) \in T_xZ$, and need to show that there exists $v \in T_xZ$ such that $\omega(u, v) \neq 0$. We look for $v = (v_1, \dots, v_n)$ of the form $v_j = iu_j + \lambda\bar\alpha_j - \bar\lambda\beta_j$, where $\lambda \in \mathbb{C}$. The condition

$$\nabla s(v) = \sum (iu_j\alpha_j + \lambda\bar\alpha_j\alpha_j - \bar\lambda\beta_j\alpha_j) + (-i\bar u_j\beta_j + \bar\lambda\alpha_j\beta_j - \lambda\bar\beta_j\beta_j) = \nabla s(Ju) + \lambda(|\alpha|^2 - |\beta|^2) = 0$$

gives $\lambda = -\dfrac{\nabla s(Ju)}{|\alpha|^2 - |\beta|^2)}$ (note that $|\alpha|^2 - |\beta|^2 \neq 0$ since $|\alpha| = |\partial s_x| > |\bar\partial s_x| = |\beta|$).

On the other hand, since $\nabla s(u) = \sum u_j\alpha_j + \bar u_j\beta_j = 0$, we have

$$\begin{aligned}\omega(u, \lambda\bar\alpha - \bar\lambda\beta) &= \operatorname{Im}(\sum \bar u_j(\lambda\bar\alpha_j - \bar\lambda\beta_j) \\ &= \operatorname{Im}(\lambda \sum \bar u_j\bar\alpha_j) - \operatorname{Im}(\bar\lambda \sum \bar u_j\beta_j) \\ &= \operatorname{Im}(\lambda \sum \bar u_j\bar\alpha_j) + \operatorname{Im}(\bar\lambda \sum u_j\alpha_j) = 0.\end{aligned}$$

Hence $\omega(u, v) = \omega(u, Ju) = |u|^2 \neq 0$.