18.966 – Homework 1 – Solutions.

1. Let E be a Lagrangian subspace of a symplectic vector space (V,Ω), and let $e_1,\dots,e_n$ be a basis of E. We proceed by induction, assuming we have constructed $f_1,\dots,f_{k-1}\in V$ such that the family $(e_1,\dots,e_n,f_1,\dots,f_{k-1})$ is free and $\Omega(e_i,f_i)=1$, $\Omega(e_i,f_j)=0$ for $i\neq j$, and $\Omega(f_i,f_j)=0$.

Because $(e_1,\dots,e_n,f_1,\dots,f_{k-1})$ is free, there exists a (non-unique) linear form $\tau\in V^*$ such that $\tau(e_i)=0$ for $i\neq k$, $\tau(f_i)=0$ for $i<k$, and $\tau(e_k)=1$. Using the fact that Ω is non-degenerate (induces an isomorphism between V and V^*), there exists $f_k\in V$ such that $\Omega(\cdot,f_k)=\tau$.

Let us check that the family $(e_1,\dots,e_n,f_1,\dots,f_k)$ is free. Indeed, if $v=\sum_{i=1}^n\lambda_ie_i+\sum_{i=1}^k\mu_if_i=0$, then $\Omega(e_i,v)=\mu_i=0$ for all $1\le i\le k$, and $v=\sum\lambda_ie_i=0$; since the (e_i) form a basis of E, we also have $\lambda_i=0$ for all i. Moreover, $\Omega(e_i,f_k)$ and $\Omega(f_i,f_k)$ are as prescribed.

Therefore, by induction we can construct $f_1,\dots,f_n$ such that $(e_1,\dots,e_n,f_1,\dots,f_n)$ is a basis of V (it's a free family and $\dim V=2n$) and the expression of Ω in this basis is the standard one.

2. S^2 is an orientable surface and hence carries a symplectic structure (its standard area form, for example); however, for $n\ge 2$, the compact manifold S^{2n} has $H^2(S^{2n},\mathbb{R})=0$, so it cannot be symplectic (for any closed 2-form, $\int_{S^{2n}}\omega^n=[\omega]^{\cup n}\cdot[S^{2n}]=0$).

The torus T^{2n} always carries a symplectic structure, induced from the standard symplectic structure of $\mathbb{R}^{2n}$ (which is preserved by translations). (On T^{2n} there are coordinates $x_1,\dots,x_n,y_1,\dots,y_n\in\mathbb{R}/\mathbb{Z}=S^1$, the symplectic form can be written as $\omega=\sum dx_i\wedge dy_i$.) Alternatively, T^{2n} is the product of n copies of T^2 which is an orientable surface. (Recall a product of symplectic manifolds is symplectic.)

3. a)
$$\begin{aligned}\iint_{[0,1]\times S^1}\Gamma^*\omega &= \int_0^1\int_{S^1}\omega_{\gamma_t(s)}\Big(\frac{\partial}{\partial t}\gamma_t(s),\frac{\partial}{\partial s}\gamma_t(s)\Big)\,ds\,dt\\ &= \int_0^1\int_{S^1}\omega_{\gamma_t(s)}(X_t(\gamma_t(s)),\dot\gamma_t(s))\,ds\,dt\\ &= \int_0^1\Big(\int_{\gamma_t} i_{X_t}\omega\Big)\,dt = \int_0^1\langle[i_{X_t}\omega],[\gamma_t]\rangle dt.\end{aligned}$$

Observing that γ_t and γ are mutually homologous (the restriction of Γ to $[0,t]\times S^1$ provides a bounding 2-chain), the r.h.s. is equal to $\int_0^1\langle[i_{X_t}\omega],[\gamma]\rangle dt=\langle\mathrm{Flux}(\rho_t),[\gamma]\rangle$.

b) Assume $\phi:(x,\xi)\mapsto(x,\xi+1)$ is generated by a time-dependent Hamiltonian vector field X_t (i.e., $\phi=\rho_1$, and $i_{X_t}\omega=dH_t$ for some Hamiltonian $H_t:M\to\mathbb{R}$). Then $\mathrm{Flux}(\rho_t)=0$ by definition ($[i_{X_t}\omega]=0$ for all t).

Recall that $\omega=d\alpha$, where $\alpha=\xi dx$, and consider the loop $\gamma:S^1\to T^*S^1$ defined by $\gamma(x)=(x,0)$, and its image $\gamma_1=\phi(\gamma)$ given by $\gamma_1(x)=(x,1)$. recall that by (1) and Stokes' theorem we have

$$\langle\mathrm{Flux}(\rho_t),[\gamma]\rangle=\iint_{[0,1]\times S^1}\Gamma^*(d\alpha)=\int_{S^1}\gamma_1^*\alpha-\int_{S^1}\gamma_0^*\alpha,$$

which implies that $\int_{\gamma_1} \alpha = \int_{\gamma_0} \alpha$, in contradiction with the direct calculation ($\int_{\gamma_0} \xi\, dx = 0$ and $\int_{\gamma_1} \xi\, dx = 2\pi$). Therefore ϕ is not Hamiltonian.

4. a) $\omega_t = \phi_t^*\omega$ is a symplectic form, and $\frac{d}{dt}\omega_t$ is an exact 1-form since it equals $\phi_t^*(L_{Y_t}\omega) = d(\phi_t^*(i_{Y_t}\omega))$ where Y_t is the vector field generating ϕ_t. Hence following Moser's argument we can find a 1-form α_t such that $d\alpha_t = -\frac{d}{dt}\omega_t$ (in this case we can e.g. take $\alpha_t = -\phi_t^*(i_{Y_t}\omega)$) and a vector field X_t such that $\alpha_t = i_{X_t}\omega_t$ (for example $X_t = -(\phi_t^{-1})_*(Y_t)$).

Let $\psi_t = \phi_t \circ \rho_t$, where ρ_t is the isotopy generated by the vector fields X_t. Then $\psi_t^*\omega = \rho_t^*(\phi_t^*\omega) = \rho_t^*\omega_t$, and

$$\frac{d(\psi_t^*\omega)}{dt} = \frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*\big(L_{X_t}\omega_t + \frac{d\omega_t}{dt}\big) = 0,$$

so ψ_t is a family of symplectomorphisms. Moreover, if we assume that the vector field X_t is tangent to Σ_0 for all t, then by integration of the differential equation $\rho_0(p) = p$, $\frac{d}{dt}\rho_t(p) = X_t(\rho_t(p))$ we obtain that ρ_t maps Σ_0 onto itself. Therefore, $\psi_t(\Sigma_0) = \phi_t(\Sigma_0) = \Sigma_t$. (Note that the flow is well-defined because M and Σ_0 are compact.)

b) Consider a point $p \in \Sigma_0$: because the symplectic orthogonal to $N_p^\omega\Sigma_0$ is exactly $T_p\Sigma_0$, the vector field X is tangent to Σ_0 at p (i.e. $X_p \in T_p\Sigma_0$) if and only if $\omega_p(X_p, v) = 0$ $\forall v \in N_p^\omega\Sigma_0$, i.e. if and only if $i_X\omega$ vanishes on $N_p^\omega\Sigma_0$.

c) Let X be a neighborhood of the zero section in $N^\omega\Sigma_0 = \{(p,v),\ p \in \Sigma_0,\ v \in N_p^\omega\Sigma_0\}$. Using e.g. the exponential map for an arbitrary metric we can construct a smooth map $\theta : X \to M$ such that $\forall p \in \Sigma_0$, $\theta(p,0) = p$, and $\forall v \in N_p^\omega\Sigma_0$, $d\theta_{(p,0)}(0,v) = v$. Consider a point $(p,0)$ of the zero section in X: we have $T_{(p,0)}X = T_p\Sigma_0 \oplus N_p^\omega\Sigma_0$, and by construction $d_{(p,0)}\theta(u,v) = u + v$ for all $u \in T_p\Sigma_0$ and $v \in N_p^\omega\Sigma_0$. However, $T_p\Sigma_0$ is a symplectic subspace of the vector space (T_pM, ω), so $T_pM = T_p\Sigma_0 \oplus N_p^\omega\Sigma_0$, and the differential of θ at p is an isomorphism. Therefore θ is a local diffeomorphism, i.e. it induces a diffeomorphism over a neighborhood U of the zero section.

At any point $p \in \Sigma_0$, the restriction to $N_p^\omega\Sigma_0$ of the 1-form $\alpha \in \Omega^1(M)$ defines a linear form $\alpha_p : N_p^\omega\Sigma_0 \to \mathbb{R}$. Let $h : N^\omega\Sigma_0 \to \mathbb{R}$ be the function defined by $h(p,v) = \alpha_p(v)$. Finally, let $\chi : N^\omega\Sigma_0 \to [0,1]$ be a smooth cut-off function equal to 1 over a neighborhood of the zero section and with support contained in U, and let $\tilde{h}(p,v) = \chi(p,v)h(p,v)$. By construction, $d_{(p,0)}\tilde{h}(0,v) = d_{(p,0)}h(0,v) = \alpha_p(v)$.

Let $f : M \to \mathbb{R}$ be the unique smooth function with support contained in $\theta(U)$ and such that $f(\theta(x)) = \tilde{h}(x)$ for all $x \in U$. Then by construction, for every $p \in \Sigma_0$ and $v \in N_p^\omega\Sigma_0$, $d_pf(v) = d_{(p,0)}\tilde{h} \circ (d_{(p,0)}\theta)^{-1}(v) = d_{(p,0)}\tilde{h}(0,v) = \alpha_p(v)$, i.e. the restriction of df to $N_p^\omega\Sigma_0$ is equal to that of α.

d) Let α_t be a smooth family of 1-forms such that $d\alpha_t = -\frac{d}{dt}\omega_t$ (for example those constructed in (a)), and let f_t be the functions constructed in (c). Then $\tilde{\alpha}_t = \alpha_t - df_t$ also satisfies the property that $d\tilde{\alpha}_t = -\frac{d}{dt}\omega_t$, and additionally the restriction of $\tilde{\alpha}_t$ to $N_p^{\omega_t}\Sigma_0$ (the orthogonal to $T_p\Sigma_0$ with respect to ω_t) vanishes at every point of Σ_0. Therefore the vector field X_t such that $i_{X_t}\omega_t = \tilde{\alpha}_t$ is tangent to Σ_0 at every point of Σ_0 (by the result of (b)), and $L_{X_t}\omega_t = -\frac{d}{dt}\omega_t$. By part (a) this completes the proof.