18.966 – Homework 2 – due Tuesday March 20, 2007.

1. Show that the sphere S^6 carries a natural almost-complex structure, induced by a vector cross-product on $\mathbb{R}^7$.

Hint: view $\mathbb{R}^7$ as the space of imaginary octonions. Octonions are the non-commutative, non-associative normed division algebra structure on $\mathbb{R}^8 = \mathbb{H} \oplus e\mathbb{H}$ with product given by the formula

$$(a + be)(a' + b'e) = (aa' - \overline{b'}b) + (b'a + b\overline{a'})e, \qquad \forall a, b, a', b' \in \mathbb{H}$$

($\overline{a'}$ is the conjugate of a', i.e. $\overline{x + yi + zj + tk} = x - yi - zj - tk$). (You may use the fact that $\|(a+be)(a'+b'e)\| = \|a+be\| \, \|a'+b'e\|$, where $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{R}^8$.)

2. Let (V, Ω) be a symplectic vector space of dimension $2n$, and let $J : V \to V$, $J^2 = -\mathrm{Id}$ be a complex structure on V.

a) Prove that, if J is Ω-compatible and L is a Lagrangian subspace of (V, Ω), then JL is also Lagrangian and $JL = L^\perp$, where $L^\perp$ is the orthogonal to L with respect to the positive inner product $g(u, v) = \Omega(u, Jv)$.

b) Deduce that J is Ω-compatible if and only if there exists a symplectic basis for V of the form

$$e_1, e_2, \ldots, e_n, f_1 = Je_1, f_2 = Je_2, \ldots, f_n = Je_n,$$

with $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ and $\Omega(e_i, f_j) = \delta_{ij}$.

3. Let (M, ω, J, g) be a symplectic manifold equipped with a compatible almost-complex structure and the corresponding Riemannian metric, and let L be a complex line bundle over M equipped with a Hermitian metric $|\cdot|$ and a Hermitian connection ∇. Given a section s of L, define $\partial s, \bar\partial s \in \Omega^1(M, L)$ by the formulas $\partial s(v) = \frac{1}{2}(\nabla s(v) - i\nabla s(Jv))$ and $\bar\partial s(v) = \frac{1}{2}(\nabla s(v) + i\nabla s(Jv))$.

It is easy to check that, $\forall x \in M$, $(\partial s)_x : T_xM \to L_x$ is $\mathbb{C}$-linear, $(\bar\partial s)_x$ is $\mathbb{C}$-antilinear, and $\nabla s = \partial s + \bar\partial s$. ($\partial s$ and $\bar\partial s$ are respectively the type (1,0) and (0,1) parts of ∇s).

a) Prove that if $(\nabla s)_x : T_xM \to L_x$ is surjective at every point x of $Z = s^{-1}(0)$, then Z is a smooth submanifold of M, and its tangent space is given by $T_xZ = \mathrm{Ker}(\nabla s)_x$.

b) Prove that, if $|\partial s| > |\bar\partial s|$ at every point of Z, then $Z = s^{-1}(0)$ is a symplectic submanifold of M. (Here $|\cdot|$ is the natural Hermitian norm on $T_x^*M \otimes L_x = \mathrm{Hom}(T_xM, L_x)$ induced by g on TM and the Hermitian metric on L).

Hint: given a point $x \in Z$, and choosing an identification between the fiber of L at x and $\mathbb{C}$ equipped with the standard norm, things essentially reduce to a linear algebra problem for the linear map $(\nabla s)_x = (\partial s)_x + (\bar\partial s)_x : T_xM \to \mathbb{C}$.