18.966 – Homework 1 – due Thursday March 1, 2007.

1. Show that, if E is a Lagrangian subspace of a symplectic vector space (V, Ω), then any basis $e_1, \dots, e_n$ of E can be extended to a standard symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ of (V, Ω).

2. For which values of n does the sphere $S^{2n} \subset \mathbb{R}^{2n+1}$ carry a symplectic structure? What about the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n} = (S^1)^{2n}$?

3. Let $\{\rho_t\}_{t\in[0,1]}$ be the isotopy generated by a time-dependent *symplectic* vector field X_t on a symplectic manifold (M, ω), i.e. $\rho_0 = \mathrm{Id}$, $\dfrac{d\rho_t}{dt} = X_t \circ \rho_t$, and $i_{X_t}\omega$ is closed. Then the *flux* of $\{\rho_t\}$ is defined to be

$$\mathrm{Flux}(\rho_t) = \int_0^1 [i_{X_t}\omega]\, dt \in H^1(M, \mathbb{R}).$$

a) Let $\gamma : S^1 \to M$ be an arbitrary closed loop, and define $\Gamma : [0,1] \times S^1 \to M$ by the formula $\Gamma(t, s) = \rho_t(\gamma(s))$, so $\gamma_t(\cdot) = \Gamma(t, \cdot)$ is the image of the loop γ by ρ_t. Prove that

$$\langle \mathrm{Flux}(\rho_t), [\gamma] \rangle = \iint_{[0,1]\times S^1} \Gamma^*\omega. \qquad (1)$$

(Remark: the right-hand side is simply the symplectic area swept by the family of loops $\{\gamma_t\}_{t\in[0,1]}$. In particular, equation (1) implies that this area depends only on the homology class represented by γ!)

b) Does the symplectomorphism $\phi : (x, \xi) \mapsto (x, \xi + 1)$ of $T^*S^1 \simeq S^1 \times \mathbb{R}$ belong to the group of Hamiltonian diffeomorphisms?

Hint: assume ϕ is generated by a Hamiltonian isotopy, and use the exactness property ($\omega = d\alpha$) to rewrite the right-hand side of equation (1) in terms of the 1-form α.

4. The goal of this problem is to prove the following result, which asserts that all deformations of compact symplectic submanifolds are induced by ambient symplectic isotopies:

Theorem 1 *Let (M, ω) be a compact symplectic manifold, and let $\{\Sigma_t\}_{t\in[0,1]}$ be a smooth family of compact symplectic submanifolds in (M, ω). Then there exists an isotopy ψ_t consisting of symplectomorphisms of M such that $\psi_t(\Sigma_0) = \Sigma_t$.*

We will admit the following classical (easy) result: there exists an isotopy consisting of *diffeomorphisms* $\phi_t : M \to M$ such that $\phi_t(\Sigma_0) = \Sigma_t$.

a) Consider $\omega_t = \phi_t^*\omega$, and prove the existence of a time-dependent vector field X_t such that $d(i_{X_t}\omega_t) = -\frac{d}{dt}\omega_t$. Show that, if the vector field X_t can be chosen to be tangent to Σ_0 at every point of Σ_0, then the theorem follows (by modifying ϕ_t by the flow of X_t).

b) Consider the symplectic normal bundle $N^\omega\Sigma_0 \subset TM_{|\Sigma_0}$ whose fiber $N^\omega_p\Sigma_0$ at a point $p \in \Sigma_0$ is the symplectic orthogonal to $T_p\Sigma_0$. Prove that the vector field X is tangent to Σ_0 if and only if, for every $p \in \Sigma_0$, the restriction to $N^\omega_p\Sigma_0$ of the 1-form $\alpha = i_X\omega$ is zero.

c) Show that, given any 1-form $\alpha \in \Omega^1(M)$ there exists a smooth function $f : M \to \mathbb{R}$ such that, for every $p \in \Sigma_0$, the restriction of df to $N^\omega_p\Sigma_0$ is equal to that of α.

d) We will admit the fact that, given one-parameter families of 1-forms α_t and of symplectic forms ω_t depending smoothly on t, the functions f_t constructed in (c) can be chosen to depend smoothly on t. Complete the proof of the Theorem.