18.965 Fall 2004
Homework 3

Due Friday 10/9/04

Exercise 1. Prove Riesz's lemma. The unit ball in a Banach space is compact if and only if the Banach space is finite dimensional.

Exercise 2. Prove that the adjoint of a compact operator is compact. Prove that if $K : X \to Y$ is compact and $T : Y \to Z$ is bounded then TK is compact.

Exercise 3. Let $L^2(S^1)$ be the set of square integrable functions on the unit circle and let $L^2_1(S^1)$ be the set of functions so that f and f' are square integrable. Show that the inclusion $L^2_1(S^1) \hookrightarrow L^2(S^1)$ is compact.

Hint: Let $f \in L^2(S^1)$ then we can expand f in a Fourier series;

$$f = \sum_n a_n e^{in\theta}.$$

and

$$\sum_n |a_n|^2 < \infty.$$

If the first derivative $f' \in L^2(S^1)$ is square integrable then

$$\sum_n n^2 |a_n|^2 < \infty.$$

Exercise 4. Suppose that a is a C^1 function on the unit circle. Using the previous exerise show the operator

$$u \mapsto iu' + au$$

is Fredholm.