18.965 Fall 2004
Homework 2

Due Monday 9/27/04

Exercise 1. Let $F : \mathbb{R} \to \mathbb{R}$ is a C^2 map with uniformly bouned first and second derivatives. F induces a map

$$\tilde{F} : C^0[0,1] \to C^0[0,1]$$

by composition; $\tilde{F}(u)$ is the function $t \mapsto F(u(t))$ Show that $\tilde{F}$ is a C^1 map. More generally let given a Banach space B let $B^0 = C^0([0,1], B)$ be the be space of contiuous maps from $[0,1]$ to B. Show that B^0 is a Banach space. If $F : B \to B$ is a C^2 map with uniformly bounded first and second derivatives, then the map induced by composition $\tilde{F}$ is C^1

Exercise 2. Let $A : B \to B$ be a bounded linear operator. Consider the linear ODE in a Banach space

$$\frac{du}{dt} + Au = 0$$

with the intial condition $u(0) = v$. First show that the solution is given by

$$e^{-tA}v$$

where the time dependent operator e^{-tA} is defined by showing the usual power series for the expotential is convergent in the Banach space of bounded linear operator from B to itself. Let $B^0 = C^0([0,\epsilon], B)$ and $B^1 = C^1([0,\epsilon], B)$. Then we can view the differntial equation as giving rise to a map

$$L : B^1 \to B^0 \times B$$

where

$$L(u) = (\frac{du}{dt}Au, u(0)).$$

Show that L is invertible and indeed its inverse is given by the familiar formula

$$L^{-1}(u,v) = e^{-tA}v + \int_0^t e^{A(s-t)}u(s)ds$$

Exercise 3. The exercise uses the previous one to prove the existence and uniqueness theorem for first order ordinary differential equations. Let B be a Banach space and let $X : B \to B$ be a C^2 map with bounded derivaitves. We seek a solution to the differntial question

$$\frac{du}{dt} + X(u) = 0$$

subject to the initial condition $u(0) = v$. Let $B^0 = C^0([0, \epsilon], B)$ and $B^1 = C^1([0, \epsilon], B)$. Then we can view the differntial equation as given rise to a map

$$F : B^1 \to B^0 \times B$$

where

$$F(u) = (\frac{du}{dt} + X(u), u(0)).$$

Assuming the first exercise show that this a C^1 map. Show that The differential at 0 is the map

$$D_0F(u) = (\frac{du}{dt} + D_0X(u), u(0))$$

which by the second exercise is invertible. Conclude from the this and the inverse function theorem the existence and unique ness theorem.

Exercise 4. Suppose that $V \to X$ is given as a subbundle of the trivial bundle $X \times \mathbb{R}^n \to X$ via a family of projections Π. Then the induced connection is $\Pi \circ d$ where d denotes the ordinary derviative. Given a local basis for V find the connection matrix for the connection. Use this formula to find a connection matrix for $\gamma \to \mathbb{CP}^n$ be the tautogical bundle. (The tautological bundle sits inside the trivial $\mathbb{C}^{n+1}$ bundle.)