18.965 Fall 04
Homework 1

Exercise 1. Prove that the grassmanians $Gr_k(\mathbb{F}^n)$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ are smooth manifolds.

Exercise 2. Prove that the $O(n)$ and $U(n)$ are smooth manifolds. Here is one hint. Show that if A is a skew symmetric (skew hermitian) matrix then

$$O = (I + A)(1 - A)^{-1}$$

is orthogonal (unitary). Thus we have map from a Euclidean space to the coresponding group. Show that this map is a homemorphism onto an open neighbhor of the identity and its inverse gives us a chart. By translating the map by elements of the group show that you get an atlas.

Exercise 3. In class we noted the coincidences of the basic smooth manifolds

$$S^1 = \mathbb{RP}^1, S^2 = \mathbb{CP}^1, S^3 = SU(2) = Sp(1), \mathbb{RP}^3 = S0(3),$$

It is also the case that $Gr_2(\mathbb{R}^3) = Gr_1(\mathbb{R}^3) = \mathbb{RP}^3$. Show that in general $Gr_k(\mathbb{F}^n)$ is diffeomorphic to $Gr_{n-k}(\mathbb{F}^n)$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}$,or $\mathbb{H}$.

Given these coincidences the obviously distinct four dimensional (compact without boundary) manifolds we know from class are

1. S^4
2. $S^3 \times S^1$
3. $S^2 \times S^2$
4. $S^2 \times \mathbb{RP}^2$
5. $S^2 \times S^1 \times S^1$
6. $S^1 \times S^1 \times S^1 \times S^1$
7. $\mathbb{RP}^4$
8. $\mathbb{RP}^3 \times S^1$
9. $\mathbb{RP}^2 \times \mathbb{RP}^2$

10. $\mathbb{RP}^2 \times S^1 \times S^1$

11. $\mathbb{CP}^2$

12. $\mathbb{HP}^1$

13. $G_2(\mathbb{R}^4)$

14. $U(2)$

Which of the manifolds in the list are diffeomorphic?