LECTURE 32: PROPERTIES OF CHERN CLASSES, THE SPLITTING PRINCIPLE

1. Axioms for Chern classes

The Chern classes $c_i(V)$ of vector bundles satisfy the following axioms.

Naturality: For V a complex vector bundle over Y, and a map $f : X \to Y$, we have
$$c_i(f^*V) = f^*c_i(V).$$
Stability: We have $c_i(V \oplus \mathbb{C}) = c_i(V)$, where $\mathbb{C}$ is the trivial bundle.
Dimension: If $\dim_{\mathbb{C}} V = n$, then $c_i(V) = 0$ for $i > n$.
Sum formula: For complex vector bundles V and W over X, we have
$$c_i(V \oplus W) = \sum_{i_1+i_2=i} c_{i_1}(V) \cup c_{i_2}(W).$$
Normalization: For the universal line bundle $L_{univ} \to \mathbb{C}P^\infty$, the first Chern class $c_1(L_{univ})$ agrees with the generator $c_1 \in H^2(\mathbb{C}P^\infty)$.

The naturality, dimension, and normalization axioms are immediate consequences of our definition of the Chern classes. Stability follows from our inductive calculation of the cohomology of $BU(n)$: analysis of the edge homomorphism tells us that the map
$$\mathbb{Z}[c_1, \dots, c_n] = H^*(BU(n)) \to H^*(BU(n-1)) = \mathbb{Z}[c_1, \dots, c_{n-1}]$$
sends c_i to c_i for $i < n$ and that c_n is mapped to zero.

We are left with the sum formula, which is the most important fact about Chern classes. We will first prove the sum formula up to a constant using the splitting principle. We will then determine the constant using the "Euler class".

2. The splitting principle

The splitting principle essentially says

> Any universal formula involving Chern classes need only be checked on sums of line bundles.

The actual statement is as follows.

Theorem 2.1 (The splitting principle)**.** Let V be an n-dimensional complex vector bundle over a space X. There exists a space $\widetilde{X}$ together with a map $f : \widetilde{X} \to X$ so that

(1) The pullback of V is given by
$$f^*(V) \cong L_1 \oplus \cdots \oplus L_n$$
where the L_i are complex line bundles over $\widetilde{X}$.

(2) The map $f^* : H^*(X) \to H^*(\widetilde{X})$ is injective.

Date: May 1, 2006.

Proof. We prove the theorem by splitting off one line at a time. Let

$$\mathbb{C}P^n \to P(V) \xrightarrow{g} X$$

be the projective space bundle given by the space

$$P(V) = \{(x, L) \,:\, x \in X, \text{ and } L \text{ is a line in the fiber } V_x\}.$$

There is a canonical line subbundle in g^*V whose fiber over a pair (x, L) is the line L. Endowing g^*V with a Hermitian structure, and letting $L^\perp$ be the orthogonal complement of L in g^*V, we have a decomposition

$$g^*V \cong L \oplus L^\perp.$$

The map f is seen to be an injection using the edge homomorphism of the cohomological Serre spectral sequence for the fibration f. There is room for non-trivial differentials if X has odd-dimensional cohomology, but these are shown to be zero using naturality and the universal example where $X = BU(n)$ and $V = V_{univ}$. $\square$