LECTURE 29: LINE BUNDLES

[The first part of this lecture was about the multiplicative structure of the Serre spectral sequence, which followed Hatcher. I also presented Hatcher's example of the cohomological Serre spectral sequence for $S^1 \to ES^1 \to \mathbb{C}P^\infty$.]

We have equivalences

$$\mathbb{C}P^\infty \simeq BU(1) \simeq K(\mathbb{Z}, 2).$$

Recall:

$$H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[c_1]$$

where $|c_1| = 2$. This generator is called the "first Chern class".

Given a space X, there are $1-1$ correspondences

$$\begin{array}{c} \{U(1)\text{-bundles over } X\} \\ \updownarrow \\ \{\text{hermitian complex line bundles over } X\} \\ \updownarrow \\ \{\text{complex line bundles over } X\}. \end{array}$$

The first correspondence associates to a principle $U(1)$-bundle P over X the line bundle

$$P \times_{U(1)} \mathbb{C} \to X$$

with a fixed hermitian structure on $\mathbb{C}$. Since every hermitian structure on $\mathbb{C}$ takes the form

$$(z, w) = az\overline{w}$$

for a a positive real, hermitian structures on a line bundle $L \to X$ are the same thing as positive values real functions on X, and any two hermitian structures on L are equivalent.

Given a line bundle L/X, there is a classifying map

$$\begin{array}{ccc} L & \longrightarrow & L_{univ} \\ \downarrow & & \downarrow \\ X & \xrightarrow[f_L]{} & BU(1) \end{array}$$

which recovers L as the pullback of L_{univ}.

Definition 0.1. The *first Chern class* $c_1(L) \in H^2(X; \mathbb{Z})$ is defined to be the class $f_L^*(c_1)$.

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Proposition 0.2. The association

$$L \mapsto c_1(L)$$

gives an isomorphism

$$\{\text{complex line bundles over } X\} \cong H^2(X; \mathbb{Z}).$$

Proof. Under the equivalence $BU(1) \simeq K(\mathbb{Z}, 2)$, c_1 gives the fundamental class of $H^2(K(\mathbb{Z}, 2), \mathbb{Z})$. □

Remark 0.3. A homework problem you were assigned indicates that if the left hand side of the isomorphism in Proposition 0.2 is given the structure of an abelian group by $\otimes$, then this is an isomorphism of groups.