LECTURE 22: THE SPECTRAL SEQUENCE OF A FILTERED COMPLEX

Let $(C_*, \{F_s C_*\}, d)$ be a filtered chain complex. We will describe an associated spectral sequence which calculates the associated graded of the homology from the homology of the associated graded chain complex:

$$E^1_{s,t} = H_{s+t}(Gr_s C_*) \Rightarrow H_{s+t}(C_*).$$

Convergence in this context means that

$$E^\infty_{s,t} = Gr_s H_{s+t}(C_*)$$

where the associated graded is taken with respect to the induced filtration on $H_*(C)$. One technical point: *we* only defined $E^\infty_{s,t}$ for spectral sequences which are eventually first quadrant.

We shall define the spectral sequence $\{E^r_{s,t}, d_r\}$ inductively with respect to r. The long exact sequences associated to the short exact sequences

$$0 \to F_{s-1}C_* \to F_s C_* \to Gr_s C_* \to 0$$

piece together to give a diagram similar to that occuring on page 3 of Hatcher's spectral sequences book. Let $F_s = F_s C_*$ and $Gr_s = Gr_s C_*$. We have already specified $E^1_{s,t}$. Let d_1 be the composite

$$d_1 : E^1_{s,t} = H_{s+t}(Gr_s) \xrightarrow{\partial} H_{s+t-1}(F_{s-1}) \to H_{s+t-1}(Gr_{s-1}) = E^1_{s-1,t}.$$

Then $E^2_{*,*}$ is necessarily the homology of $(E^1_{*,*}, d_1)$. Assume that $E^r_{s,t}$ has been defined. We need to define d_r. Suppose that $[x] \in E^r_{s,t}$ is represented by $x \in E^1_{s,t} = H_{s+t}(Gr_s)$. Consider the diagram

$$\begin{array}{ccccc}
 & & H_{s+t-1}(F_{s-r}) & \longrightarrow & H_{s+t-1}(Gr_{s-r}) \\
 & & \downarrow & & \\
 & & H_{s+t-1}(F_{s-r+1}) & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
H_{s+t}(Gr_s) & \xrightarrow[\partial]{} & H_{s+t-1}(F_{s-1}) & &
\end{array}$$

The differential d_r is given by

$$d_r([x]) = [y]$$

where $[y] \in E^r_{s-r,t+r-1}$ is represented by $y \in E^1_{s-r,t+r-1} = H_{s+t-1}(Gr_{s-r})$, and the element y is given by the following process:

(1) send x to $\partial x \in H_{s+t-1}(F_{s-1})$.

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(2) lift ∂x to $\widetilde{\partial x} \in H_{s+t-1}(F_{s-r})$.

(3) take y to be the image of $\widetilde{\partial x}$ in $H_{s+t-1}(Gr_{s-r})$.

There are many things to check, such as that the lift $\widetilde{\partial x}$ exists, that d_r is independent of the choice of lift, that d_r is independent of the choice of representative x, and that $d_r^2 = 0$. These are all inductive diagram chases involving the various long exact sequences. Having done this, we can define $E^{r+1}_{*,*}$ to be the homology of $(E^r_{*,*}, d_r)$.

We now sketch the origin of the isomorphism

$$\phi : E^{\infty}_{s,t} \cong Gr_s H_{s+t}(C).$$

An element $[z] \in E^{\infty}_{s,t}$ is represented by an element $z \in E^1_{s,t} = H_{s+t}(Gr_s)$ for which $d_r([z]) = 0$ for all r. Inductively, this means that ∂z lifts to $H_{s+t-1}(F_{s-r})$ for r arbitrarily large. Since $F_s = 0$ for $s < 0$, we may conclude that $\partial r = 0$. We conclude that z is in the image of the map

$$H_{s+t}(F_s) \to H_{s+t}(Gr_s).$$

Let w be an element that maps to z. Take $\overline{w}$ to be the image of w in $Gr_s H_{s+t}(C)$. Then $\phi([z]) = \overline{w}$. Again, many things need to be checked, but this can be a fun activity.