LECTURE 12: HOMOTOPY EXCISION

Unlike homology, homotopy groups do not satisfy excision. If they did, then the suspension would induce an isomorphism

$$\Sigma : \pi_k(X) \xrightarrow{\cong ?} \pi_k(\Sigma X).$$

This cannot happen: we have seen that $\pi_2(S^1) = 0$ while $\pi_3(S^2) = \mathbb{Z}$. However, excision does hold through a range.

Theorem 0.1 (Homotopy excision)**.** Suppose that $f : A \to X$ is an m-equivalence and $g : A \to Y$ is an n-equivalence. Assume that one of these maps is the inclusion of a relative CW-complex. Let Z be the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow \\ Y & \xrightarrow[f']{} & Z \end{array}$$

Then the induced map of homotopy fibers

$$F(f) \to F(f')$$

is an $(n + m - 1)$-equivalence.

Proofs of related theorems may be found in Hatcher or May.

Definition 0.2. A map of pairs $f : (X, A) \to (Y, B)$ is an *n-equivalence* if the induced map on relative homotopy groups

$$f_* : \pi_k(X, A) \to \pi_k(Y, B)$$

is an isomorphism for $k < n$ and an epimorphism for $k = n$.

Specializing to the case where (X, A) and (Y, A) are relative CW-complexes, we get a form of the homotopy excision theorem which bears a more close resemblance to what you might think of by "excision".

Corollary 0.3. Suppose that (X, A) and (Y, A) are relative CW complexes, that the inclusion $A \hookrightarrow X$ is an m-equivalence, and that the inclusion $A \hookrightarrow Y$ is an n-equivalence. Let $Z = X \cup Y$. Then the natural map

$$(X, A) \to (Z, Y)$$

is an $(m + n)$-equivalence.

Specializing to the case of a mapping cone, we obtain

Corollary 0.4. Suppose that A is an $(m - 1)$-connected CW-complex and that $f : A \to X$ is an n-equivalence. Then the natural map

$$F(f) \to \Omega C(f)$$

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is an $(m+n-1)$-equivalence. In particular, if (X, A) is a relative CW-complex, then the map of pairs

$$(X, A) \to (X/A, *)$$

is an $(m+n)$-equivalence.

Specializing the corollary above to the case where $X = *$, we recover the Freudenthal suspension theorem.

Corollary 0.5 (Freudenthal suspension theorem)**.** Suppose that A is $(m-1)$-connected. Then the suspension

$$A \to \Omega\Sigma A$$

is a $(2m-1)$-equivalence.

As a result we deduce that the map

$$\pi_k(S^m) \to \pi_{k+1}(S^{m+1})$$

is an epimorphism if $k = 2m-1$ and is an isomorphism if $k < 2m-1$. The Hopf fibration allowed us to deduce that $\pi_2(S^2) \cong \mathbb{Z}$. We therefore have

Corollary 0.6. The group $\pi_n(S^n)$ is isomorphic to $\mathbb{Z}$, generated by the identity map.