LECTURE 9: FIBRATIONS

Definition 0.1. A map $f : X \to Y$ is a *fibration* if it satisfies the covering homotopy property (CHP): for all maps f and homotopies H making the square commute

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{f} & X \\ \downarrow & \overset{\widetilde{H}}{\nearrow} & \downarrow f \\ Z \times I & \xrightarrow[H]{} & Y \end{array}$$

there exists a lift $\widetilde{H}$ making the diagram commute.

We saw that cofibrations $f : X \to Y$ had the property that the canonical map

$$C(f) \to Y/X$$

is a homotopy equivalence. On the homework, you will verify:

Lemma 0.2. Suppose that $f : X \to Y$ is a fibration, and suppose that Y is pointed. The canonical map

$$f^{-1}(*) \to F(f)$$

is a homotopy equivalence.

The infinite fiber sequence therefore yields the following corollary:

Corollary 0.3. For a fibration $f : X \to Y$ with fiber $F = f^{-1}(*)$, there is a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(X) \xrightarrow{f_*} \pi_n(Y) \to \pi_{n-1}(F) \to \cdots$$

Examples of fibrations:

(1) Covering spaces: the CHP is easily obtained from the homotopy lifting properties of covering spaces.
(2) Products: a projection $X \times F \to X$ is easily seen to be a fibration.
(3) Locally trivial bundles: a map $f : X \to Y$ is a locally trivial bundle with fiber F if there is an open cover $\{U_i\}$ of Y such that there are homeomorphisms $f^{-1}(U_i) \approx U_i \times F$. If Y is paracompact, then f is a fibration.

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