LECTURE 5: COFIBRATIONS, WELL POINTEDNESS, WEAK EQUIVALENCES, RELATIVE HOMOTOPY

In other words, today's lecture consisted of a hodgepodge of odds and ends.

1. Cofibrations and well pointedness

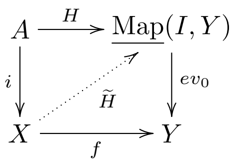
If $i : A \hookrightarrow X$ is an inclusion of a subcomplex into a CW complex, then there is an isomorphism

$$H^n(X, A) \cong \widetilde{H}^n(X/A).$$

This may not hold for general subspaces A in X. We abstract a property that we will later see makes this true.

Let $ev_0 : \underline{\mathrm{Map}}(I, Y) \to Y$ be the "evaluation at 0" map.

Definition 1.1. A map $i : A \to X$ is a *cofibration* if it satisfies the homotopy extension property (HEP): for each map $f : X \to Y$, and each homotopy $H : A \to \underline{\mathrm{Map}}(I, Y)$ making the square commute:



there exists an extension homotopy $\widetilde{H}$ making the upper and lower triangles commute.

Remark 1.2. It turns out that a cofibration is necessarily an inclusion with closed image. Being a cofibration is equivalent to being an neighborhood deformation retract (NDR) pair (see May). This roughly means that the is a neighborhood of A in X for which A is a deformation retract (the actual definition is more complicated). Thus it is common for closed inclusions to be cofibrations.

Definition 1.3. A space $X \in \mathrm{Top}_*$ is *well-pointed* if the inclusion $* \hookrightarrow X$ is a cofibration.

Let $\mathrm{Susp}(X)$ be the unreduced suspension. It is the space obtained from $X \times I$ by collapsing the ends of the cylinder.

In the homework problem where I asked you to show $\widetilde{H}^n(X) \cong \widetilde{H}^{n+1}(\Sigma X)$ I should have assumed that X was well pointed. I am assigning the following in the next homework.

Lemma 1.4. Suppose that X is well pointed. Then the quotient map

$$\mathrm{Susp}(X) \to \Sigma X$$

is a homotopy equivalence.

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Not every pointed space is well pointed. However, if a pointed space X is not well pointed, we can form a new "whiskered" space $X_w = X \cup_{\{0\}} I$ where we glue an interval to the basepoint. We give X_w the basepoint $\{1\}$. You will verify:

- The inclusion $X \hookrightarrow X_w$ is a deformation retract.
- X_w is well pointed.

2. Weak equivalences

The action of the fundamental groupoid on the higher homotopy groups is described by a functor

$$\pi_k(X, -) : \pi_{oid}(X) \to \text{Groups}.$$

In particular, because $\pi_{oid}(X)$ is a groupoid, a path γ from x to y must induce an isomorphism

$$\gamma_* : \pi_k(X, x) \to \pi_k(X, y).$$

Definition 2.1. A map of spaces $f : X \to Y$ is a *weak homotopy equivalence*, or simply a *weak equivalence* if

(1) $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection.
(2) $f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism for all $k > 0$ and all $x \in X$.

We used the action of the fundamental groupoid to prove the following proposition.

Proposition 2.2. Homotopy equivalences are weak homotopy equivalences.

3. Relative homotopy groups

Let X be pointed, and let A be a subspace of X containing the basepoint. We define relative homotopy groups

$$\pi_k(X, A) = [(I^k, \partial I^k, \partial I^k - (I^{k-1} \times \{0\})), (X, A, *)].$$

That is, maps of the k-cube which send the boundary into A, and which sent all but one of the faces of the cube to the basepoint, up to homotopies which preserve these conditions.

For $k = 0$, relative homotopy is not defined. For $k = 1$, relative homotopy is a set. For $k \geq 2$, relative homotopy is a group, with the group operation given by juxtaposition of cubes. For $k \geq 3$, these groups are abelian.

Much like relative homology, relative homotopy fits into a long exact sequence:

$$\begin{aligned} &\cdots \to \pi_k(A) \xrightarrow{i_*} \pi_k(X) \xrightarrow{j_*} \pi_k(X, A) \\ &\xrightarrow{\partial} \pi_{k-1}(A) \to \cdots \\ &\cdots \to \pi_1(X, A) \to \pi_0(A) \to \pi_0(X) \end{aligned}$$

The end of this sequence must be interpreted appropriately, because these are just sets: $\pi_1(X)$ acts on $\pi_1(X, A)$, with orbits given by the subset of $\pi_0(A)$ sent to the basepoint component in $\pi_0(X)$.