LECTURE 3: POINTED SPACES AND HOMOTOPY GROUPS

1. Pointed spaces

Let Top_* denote the category of pointed spaces. The objects are pairs (X, x) where $X \in \mathrm{Top}$ and x is a point of X. The morphisms are base-point preserving continuous maps.

Let X, Y be objects of Top_*. The space of pointed maps is given as the subspace of the mapping space consisting of the basepoint preserving maps:

$$\underline{\mathrm{Map}}_*(X, Y) \subseteq \underline{\mathrm{Map}}(X, Y).$$

Note that $\underline{\mathrm{Map}}_*(X, Y)$ is itself pointed, with basepoint given by the constant map.

The topological adjunction between $- \times -$ and $\underline{\mathrm{Map}}(-, -)$ has a pointed analog.

Proposition 1.1. The natural map gives rise to a homeomorphism

$$\underline{\mathrm{Map}}_*(X \wedge Y, Z) \approx \underline{\mathrm{Map}}_*(X, \underline{\mathrm{Map}}_*(Y, Z)).$$

This homeomorphism is natural in X, Y, and Z.

One could view this as motivation for the smash product construction.

Given an unpointed space X, one can make a pointed space X_+ by adding a disjoint basepoint. We have adjoint functors

$$(-)_+ : \mathrm{Top} \leftrightarrows \mathrm{Top}_* : \mathrm{forget}.$$

We have the following useful identities.

$$S^1 \wedge S^n \approx S^{n+1}$$
$$X \wedge S^0 \approx X$$
$$X_+ \wedge Y_+ \approx (X \times Y)_+$$

Let $f, g : X \to Y$ be pointed maps. A pointed homotopy $H : f \simeq g$ may be conveniently represented by a map

$$H : X \wedge I_+ \to Y.$$

Let $[X, Y]_*$ be the set of pointed maps modulo pointed homotopy.

Lemma 1.2. There is a bijection

$$\pi_0 \underline{\mathrm{Map}}_*(X, Y) \cong [X, Y]_*.$$

There are two useful adjoint functors

$$\Sigma : \mathrm{Top}_* \leftrightarrows \mathrm{Top}_* : \Omega$$

given by

$$\Sigma X = X \wedge S^1$$
$$\Omega X = \underline{\mathrm{Map}}_*(S^1, X)$$

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Note that there are homeomorphisms $\Sigma S^n \approx S^{n+1}$.

The disjoint union of unpointed spaces satisfies a universal property:

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow^{f} & \\ X \amalg Y & \xrightarrow[\exists!]{f \amalg g} & Z \\ \uparrow & \nearrow_{g} & \\ Y & & \end{array}$$

For pointed spaces X, Y, define the wedge to be the union where we have identified the basepoints

$$X \vee Y = X \cup_* Y.$$

The wedge satisfies a similar universal property in the category of pointed spaces. For $X, Y, Z \in \text{Top}_*$ we have

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow^{f} & \\ X \vee Y & \xrightarrow[\exists!]{f \vee g} & Z \\ \uparrow & \nearrow_{g} & \\ Y & & \end{array}$$

2. Homotopy groups

Let X be a pointed space. Define

$$\pi_n(X) = [S^n, X]_*$$

For $n \geq 1$ this is a group. Let

$$\text{pinch} : S^n \to S^n \vee S^n$$

be the pinch map (identify all elements on the equator). For $\alpha, \beta : S^n \to X$, define $\alpha + \beta$ to be the composite

$$\alpha + \beta : S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{\alpha \vee \beta} X$$

Lemma 2.1. This binary operation descends to give a group structure on $\pi_n(X)$.

Alternatively, addition may be described on maps $(I^n, \partial I^n) \to (X, *)$ by juxtaposition of cubes. See Hatcher or May for this perspective. This perspective gives an easy geometric proof of the following

Proposition 2.2. For $k > 1$, $\pi_k(X)$ is abelian.

Given a pointed map $f : X \to Y$, there is an induced map

$$f_* : \pi_k(X) \to \pi_k(Y)$$
$$[\alpha] \mapsto [f \circ \alpha].$$

Thus the homotopy groups form a functor

$$\pi_k(-) : \text{Top}_* \to \text{Groups}.$$