HOMEWORK 1

DUE MONDAY, FEB 13

Here are a couple of exercises in category theory, to acclimate you with the definitions.

1. *Yoneda lemma.* Let $\mathcal{C}$ be a category. We may consider the category $\text{Funct}(\mathcal{C}^{op}, \textit{Sets})$ whose objects are contravariant functors $\mathcal{C} \to \textit{Sets}$ and whose morphisms are natural transformations, ignoring the caveat that the collection of natural transformations between two functors may not form a set. We have seen that objects $Z \in \mathcal{C}$ give rise to contravariant functors

$$F_Z : \mathcal{C} \to \textit{Sets}$$
$$X \mapsto \text{Map}_{\mathcal{C}}(X, Z) = F_Z(X).$$

We have also seen that morphisms $f : Z_1 \to Z_2$ give rise to natural transformations

$$f_* : F_{Z_1} = \text{Map}_{\mathcal{C}}(-, Z_1) \to \text{Map}_{\mathcal{C}}(-, Z_2) = F_{Z_2}.$$

We thus have a functor

$$\mathcal{Y} : \mathcal{C} \to \text{Funct}(\mathcal{C}, \textit{Sets})$$

given by $Y(Z) = F_Z$. This functor is called the *Yoneda embedding*.

Prove *Yoneda's lemma*: the map

$$\text{Map}_{\mathcal{C}}(Z_1, Z_2) \to \text{Nat}(F_{Z_1}, F_{Z_2})$$

is a bijection. Here, $\text{Nat}(F_{Z_1}, F_{Z_2})$ is the collection of natural transformations. In particular, F_{Z_1} and F_{Z_2} are naturally isomorphic functors if and only if Z_1 and Z_2 are isomorphic.

2. *Adjoint functors.* Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A pair of covariant functors

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

are said to form an *adjoint pair* (F, G) if there is a natural isomorphism

$$\eta : \text{Map}_{\mathcal{D}}(F(-), -) \xrightarrow{\cong} \text{Map}_{\mathcal{C}}(-, G(-))$$

between functors from $\mathcal{C}^{op} \times \mathcal{D} \to \textit{Sets}$. Such an isomorphism η is called an *adjunction.* We say that F is *left adjoint* to G, and that G is *right adjoint* to F.

- **(a):** Show that if G' is also right adjoint to F, then there is a natural isomorphism $G \cong G'$ (hint: you can use the Yoneda lemma).
- **(b):** Show that if F' is also left adjoint to G, then there is a natural isomorphism $F \cong F'$ (hint: deduce this from (a) by being sneaky).
- **(c):** Let S be a set. Show that there is an adjunction

$$\text{Map}(X \times S, Y) \cong \text{Map}(X, \text{Map}(S, Y)).$$