Throughout this lecture k denotes an algebraically closed field.

17.1 Tangent spaces and hypersurfaces

For any polynomial $f \in k[x_1, \ldots, x_n]$ and point $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$ we define the affine linear form

$$f_P(x_1, \ldots, x_n) \;:=\; \sum_{i=1}^n \frac{\partial f}{\partial x_i}(P)(x_i - a_i).$$

The zero locus of f_P in $\mathbb{A}^n$ is an *affine hyperplane* in $\mathbb{A}^n$, a subvariety isomorphic to $\mathbb{A}^{n-1}$. Note that $f_P(P) = 0$, so the zero locus contains P.

Definition 17.1. Let P be a point on an affine variety V. The *tangent space* of V at P is the variety $T_P(V)$ defined by the ideal $\{f_p : f \in I(V)\}$.

It is clear that $T_p(V)$ is a variety; indeed, it is the nonempty intersection of a set of affine hyperplanes in $\mathbb{A}^n$ and therefore an affine subspace of $\mathbb{A}^n$ isomorphic to $\mathbb{A}^d$, where $d = \dim T_P(V)$. Note that the definition of $T_P(V)$ does not require us to choose a set of generators for $I(V)$, but for practical applications we want to be able to compute $T_P(V)$ in terms of a finite set of generators for $I(V)$. The following lemma shows that we can do this, and, most importantly, it does not matter which set of generators we pick.

Lemma 17.2. *Let P be a point on an affine variety V. If $f_1, \ldots, f_m$ generate $I(V)$, then the corresponding affine linear forms $f_{1,P}, \ldots, f_{m,P}$ generate $I(T_P(V))$.*

Proof. Let $g = \sum h_i f_i$ be an element of $I(V)$. Applying the product rule and the fact that $f_{i,P}(P) = 0$ yields

$$g_P = \sum_i \bigl(h_i(P) f_{i,P} + h_{i,P} f_i(P)\bigr) = \sum_i h_i(P) f_{i,P}, \tag{1}$$

which is an element of the ideal $(f_{1,P}, \ldots, f_{m,P})$. Thus $I(T_p(V)) = (f_{1,P}, \ldots, f_{m,P})$. □

When considering the tangent space of a variety at a particular point P, we may assume without loss of generality that $P = (0, \ldots, 0)$, since we can always translate the ambient affine space $\mathbb{A}^n$; this is just a linear change of coordinates (indeed, this is the very definition of affine space, it is a vector space without a distinguished origin). We can then view the affine subspace $T_P(V) \subseteq \mathbb{A}^n$ as a linear subspace of the vector space k^n. The affine linear forms f_P are then linear forms on k^n, equivalently, elements of the dual space $(k^n)^\vee$.

Recall from linear algebra that the dual space $(k^n)^\vee$ is the space of linear functionals $\lambda\colon k^n \to k$. The orthogonal complement $S^\perp \subseteq (k^n)^\vee$ of a subspace $S \subseteq k^n$ is the set of linear functionals λ for which $\lambda(P) = 0$ for all $P \in S$; it is a subspace of $(k^n)^\vee$, and since k^n has finite dimension n, we have $\dim S + \dim S^\perp = n$.

Theorem 17.3. *Let P be a point on an affine variety $V \subseteq \mathbb{A}^n$ with ideal $I(V) = (f_1, \ldots, f_m)$. If we identify $\mathbb{A}^n$ with the vector space k^n with origin at P, the subspace of $(k^n)^\vee$ spanned by the linear forms $f_{1,P}, \ldots, f_{m,P}$ is $T_P(V)^\perp$, the orthogonal complement of $T_p(V)$.*

Andrew V. Sutherland

Proof. This follows immediately from Lemma 17.2 and its proof; the set of linear forms in $I(T_P(V))$ is precisely the set of linear forms that vanish at every point in $T_p(V)$, which, by definition, is the orthogonal complement $T_P^\vee$. Moreover, we see from (1) that every linear form in $I(T_P(V))$ is a k-linear combination of $f_{1,P}\ldots,f_{m,P}$. $\square$

The vector space $T_P(V)^\perp$ is called the *cotangent space* of V at P. As noted above, as a variety, $T_P(V)$ is isomorphic to some $\mathbb{A}^d$, where $d=\dim T_P(V)$, and it follows that the dimenstion of $T_P(V)$ as a vector space is the same as its dimension as a variety, since $\dim \mathbb{A}^d = d = \dim_k k^d$. The dimension of $T_P(V)^\perp$ is then $n-d$.

Recall from Lecture 13 the Jacobian matrix

$$J_P = J_P(f_1,\ldots,f_m) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(P) & \cdots & \frac{\partial f_1}{x_m}(P) \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(P) & \cdots & \frac{\partial f_n}{x_m}(P) \end{pmatrix}.$$

For a variety V with $I(V)=(f_1,\ldots,f_m)$, we defined a point $P\in V$ to be *smooth* (or nonsingular) precisely when $\operatorname{rank} J_P = n-\dim V$. Viewing J_P as the matrix of a linear transformation from $(k^n)^\vee$ to $(k^n)^\vee$ whose image is $T_P(V)^\perp$, we obtain the following corollary of Theorem 17.3.

Corollary 17.4. *Let P be a point on an affine variety $V\subseteq\mathbb{A}^n$ with $I(V)=(f_1,\ldots,f_m)$, and let $J_P=J_P(f_1,\ldots,f_m)$. Then $\dim T_P(V)^\perp = \operatorname{rank} J_P$ and $\dim T_P(V) = n-\operatorname{rank} J_P$. In particular, the rank of J_P does not depend on the choice of generators for $I(V)$ and P is a smooth point of V if and only if $\dim T_P = \dim V$.*

Remark 17.5. For projective varieties V we defined smooth points P as points that are smooth in all (equivalently, any) affine part containing P. One can also define tangent spaces and Jacobian matrices for projective varieties directly using generators for the homogeneous ideal of V. This is often more convenient for practical computations.

Corollary 17.14 makes it clear that, as claimed in Lecture 13, our notion of a smooth point $P\in V$ is well defined; it does not depend on which generators $f_1,\ldots,f_m$ of $I(V)$ we use to compute J_P, or even on the number of generators. Now we want to consider what can happen when $\dim T_P(V)\neq\dim V$. Intuitively, we would should expect that $\dim T_P(V)$ is then strictly greater than $\dim V$; this is easy to see when V is defined by a single equation, since then $J_P(f)$ has just one row and its rank is either 0 or 1. We will prove that we always have $\dim T_P(V)\geq\dim V$ by reducing to this case.

Definition 17.6. A variety V for which $I(V)$ is a nonzero principal ideal is a *hypersurface*.

Lemma 17.7. *Every hypersurface in $\mathbb{A}^n$ or $\mathbb{P}^n$ has dimension $n-1$.*

Proof. Let $V\subseteq\mathbb{A}^n$ be a hypersurface with $I(V)=(f)$ for some nonzero $f\in k[x_1,\ldots,x_n]$. We must have $\dim V\leq n-1$, since $V\subsetneq\mathbb{A}^n$. Let $\phi\colon k[x_1,\ldots,x_n]\to k[x_1,\ldots,x_n]/(f)$ be the quotient map. We must have $f\notin k$, since $V\neq\emptyset$, so $\deg_{x_i} f>0$ for some x_i, say x_1. If $\dim V<n-1$ then the transcendence degree of $k(V)$ is less than $n-1$, therefore $\phi(x_2),\ldots,\phi(x_n)$ must be algebraically dependent as elements of $k(V)$. Thus there exists $g\in k[x_2,\ldots,x_n]$ such that $g(\phi(x_2),\ldots,\phi(x_n))=0$. But then $\phi(g)=0$, so $g\in\ker\phi=(f)$. But this is a contradiction, since $\deg_{x_1} g=0$. So $\dim V=n-1$. If $V\subseteq\mathbb{P}^n$, then one of its affine parts V_i is a hypersurface in $\mathbb{A}^n$, and then $\dim V=\dim V_i=n-1$. $\square$

The converse to Lemma 17.7 is true; every variety of codimension 1 is a hypersurface. This follows from the general fact that every variety is birationally equivalent to a hypersurface. Recall that a function field F/k if any finitely generated extension; the *dimension* of a function field is its transcendence degree.

Theorem 17.8. *Let F/k be a function field of dimension n. Then there exist algebraically independent elements $\alpha_1, \ldots, \alpha_n \in F$ and an element α_{n+1} algebraic over $k(\alpha_1, \ldots, \alpha_m)$ such that $F = k(\alpha_1, \ldots, \alpha_{n+1})$.*

The following proof is adapted from [1, App. 5, Thm. 1].

Proof. Let $\gamma_1, \ldots, \gamma_m$ be a set of generators for F/k of minimal cardinality m, ordered so that $\gamma_1, \ldots, \gamma_n$ is a transcendence basis (every set of generators contains a transcendence basis). If $m = n$ then we may take $\gamma_{n+1} = 0$ and we are done. Otherwise γ_{n+1} is algebraic over $k(\gamma_1, \ldots, \gamma_n)$, and we claim that in fact $m = n + 1$ and we are also done.

Suppose $m > n + 1$. Let $f \in k[x_1, \ldots, x_{n+1}]$ be irreducible with $f(\gamma_1, \ldots, \gamma_{n+1}) = 0$; such an f exists since $\gamma_1, \ldots, \gamma_{n+1}$ are algebraically dependent. We must have $\partial f/\partial x_i \neq 0$ for some x_i; if not than we must have $\operatorname{char}(k) = p > 0$ and $f = g(x_1^p, \ldots, x_{n+1}^p) = g^p(x_1, \ldots, x_{n+1})$ for some $g \in k[x_1, \ldots, x_{n+1}]$, but this is impossible since f is irreducible. It follows that γ_i is algebraic, and in fact separable, over $K = k(\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_{n+1})$; the irreducible polynomial $f(\gamma_1, \ldots, \gamma_{i-1}, x_i, \gamma_{i+1}, \ldots, \gamma_{n+1|})$ has γ_i as a root, and its derivative is nonzero. Now γ_m is also algebraic over K, and it follows from the primitive element theorem [2, §6.10] that $K(\gamma_i, \gamma_m) = K(\delta)$ for some $\delta \in K$.[1] But this contradicts the minimality of m, so we must have $m = n + 1$ as claimed. □

Remark 17.9. Theorem 17.8 holds for any perfect field k; it is not necessary for k to be algebraically closed.

Theorem 17.10. *Every affine (resp. projective) variety of dimension n is birationally equivalent to a hypersurface in $\mathbb{A}^{n+1}$ (resp. $\mathbb{P}^{n+1}$).*

Proof. Two projective varieties are birationally equivalent if and only if all their nonempty affine parts are, and the projective closure of a hypersurface is a hypersurface, so it suffices to consider affine varieties. Recall from Lecture 15 that varieties are birationally equivalent if and only if their function fields are isomorphic, and it follows from Theorem 17.8 that every function field arises as the function field of a hypersurface: if $k(V) = k(\gamma_1, \ldots, \gamma_{n+1})$ with $\gamma_1, \ldots, \gamma_n)$ algebraically independent, then there exists an irreducible polynomial f in $k[x_1, \ldots, x_{n+1}]$ for which $f(\gamma_1, \ldots, \gamma_{n+1}) = 0$, and then V is birationally equivalent to the zero locus of f in $\mathbb{A}^{n+1}$. □

Corollary 17.11. *For any point P on an affine variety V we have $\dim T_P(V) \geq \dim V$.*

Corollary 17.12. *The set of singular points of a variety is a closed subset; equivalently, the set of nonsingular points is a dense open subset.*

Proof. It suffices to prove this for affine varieties. So let $V \subseteq \mathbb{A}^n$ be an affine variety with ideal $(f_1, \ldots, f_m)$, and for any $P \in V$ let $J_P = J_P(f_1, \ldots, f_m)$ be the Jacobian matrix. Then

$$\mathrm{Sing(V) := \{P\colon\ \dim T_P(V) > \dim V\} = \{P\colon\ rank\, J_P < n - \dim V\}}$$

[1] As noted in [2], to prove $K(\alpha, \beta) = K(\delta)$ for some $\delta \in K(\alpha, \beta)$, we only need one of α, β to be separable.

is the set of singular points on V. Let $r = n - \dim V$. We have rank $J_P < r$ if and only if every $r \times r$ minor of J_P has determinant zero. If we now consider the matrix of polynomials $(\partial f_i/\partial x_j)$, the determinant of each of its $r \times r$ minors is a polynomial in $k[x_1, \ldots, x_n]$, and Sing(V) is the intersection of V with the zero locus of all these polynomials. Thus Sing(V) is an algebraic set, hence closed. $\square$

Recall the one-to-one correspondence between points $P = (a_1, \ldots, a_n)$ in $\mathbb{A}^n$ and maximal ideals $M_P = (x_1 - a_1, \ldots, x_n - a_n)$ of $k[\mathbb{A}^n]$. If $V \subseteq \mathbb{A}^n$ is an affine variety, then the maximal ideals m_P of its coordinate ring $k[V] = k[\mathbb{A}^n]/I(V)$ are in one-to-one correspondence with the maximal ideals M_P of $k[\mathbb{A}^n]$ that contain $I(V)$; these are precisely the maximal ideals M_P for which $P \in V$.

If we choose coordinates so that $P = (0, \ldots, 0)$, then M_P is a k-vector space that contains M_P^2 as a subspace, and the quotient space M_P/M_P^2 is then also a k-vector space. Indeed, its elements correspond to (cosets of) linear forms on k^n. We may similarly view m_P, m_P^2, and m_P/m_P^2 as k-vector spaces, and this leads to the following theorem.

Theorem 17.13. *Let P be a point on an affine variety V. Then $T_P(V)^\vee \simeq m_P/m_P^2$.*

Proof. As above we assume without loss of generality that $P = (0, \ldots, 0)$. Then M_P consists of the polynomials in $k[x_1, \ldots, x_n]$ for which each term has degree at least 1 (equivalently, constant term 0). We now consider the linear transformation

$$D\colon M_P \to (k^n)^\vee$$

that sends $f \in M_P$ to the linear form $f_P \in (k^n)^\vee$. This map is surjective, and its kernel is M_P^2; we have $f_P = 0$ if and only if $\partial f/\partial x_i(0) = 0$ for $i = 1, \ldots, n$, and this occurs precisely when every term in f has degree at least 2, equivalently, $f \in M_P^2$. It follows that

$$M_P/M_P^2 \simeq (k^n)^\vee.$$

The restriction map $(k^n)^\vee \to (T_P)^\vee$ that restricts the domain of a linear form on k^n to $T_P(V)$ is surjective, and composing this with D yields a surjective linear transformation

$$d\colon M_P \to T_P(V)^\vee$$

whose kernel we claim is equal to $M_P^2 + I(V)$ (this is a sum of ideals in $k[x_1, \ldots, x_n]$ that is clearly a subset of M_P). A polynomial $f \in M_P$ lies in $\ker d$ if and only if the restriction of f_P to $T_P(V)$ is the zero function, which occurs if and only if $f_P = g_P$ for some $g \in I(V)$, since T_P the zero locus of g_P for $g \in I(V)$. But this happens if and only if $f - g$ lies $\ker D = M_P^2$, equivalently, $f \in M_P^2 + I(V)$.

We therefore have

$$T_P(V)^\vee \simeq \frac{M_P}{M_P^2 + I(V)} \simeq \frac{M_P/I(V)}{(M_P^2 + I(V)/I(V)} = \frac{M_P/I(V)}{M_P^2/I(V)} \simeq m_P/m_P^2. \qquad \square$$

Corollary 17.14. *The smooth points P on a variety V are precisely the points P for which*

$$\dim m_P/m_P^2 = \dim V = \dim k[V]$$

The three dimensions in the corollary above are, respectively, the dimension of m_P/m_P^2 as a k-vector space, the dimension of V as a variety, and the Krull dimension of the coordinate ring $k[V]$; as noted in Lecture 13, we always have $\dim V = \dim k[V]$. The key point is that we now have a completely algebraic notion of smooth points. If R is any affine algebra, the maximal ideals $\mathfrak{m}$ of R correspond to smooth points on a variety with coordinate ring R, and we can characterize the "smooth" maximal ideals as those for which $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$, where $k = R_{\mathfrak{m}}/\mathfrak{m}$ is now the residue field of the localization of R at $\mathfrak{m}$. Smooth varieties then correspond to affine algebras R in which every maximal ideal is "smooth".

References

[1] I. R. Shafarevich, *Basic algebraic geometry*, 2nd edition, Springer-Verlag, 1994.

[2] B. L. van der Waerdan, *Algebra, Volume I*, 7th edition, Springer, 1991.