## FORMULARIUM FOR DIVISOR CLASSES

**Exercise 0.1.** Let

$$\pi_{n+1} : \overline{\mathrm{M}}_{g,n+1} \to \overline{\mathrm{M}}_{g,n}$$

be the morphism that forgets the $n+1$st marked point. Prove the following formulae:

(1) $\pi_{n+1}^*(\kappa) = \kappa - \psi_{n+1}$.

(2) $\pi_{n+1}^*(\psi_i) = \psi_i - \delta_{0,\{i,n+1\}}$ for $i \leq n$.

(3) $\pi_{n+1}^*(\delta_{irr}) = \delta_{irr}$.

(4) $\pi_{n+1}^*(\delta_{h,S}) = \delta_{h,S} + \delta_{h,S\cup\{n+1\}}$.

**Exercise 0.2.** Let

$$\xi : \overline{\mathrm{M}}_{g-1,n\cup\{x,y\}} \to \overline{\mathrm{M}}_{g,n}$$

be the morphism that glues the two points $x, y$. Show that $\xi$ pulls back the tautological classes as follows:

(1) $\xi^*(\kappa) = \kappa$.

(2) $\xi^*(\phi_i) = \phi_i$ for $i \leq n$.

(3) $\xi^*(\delta_{irr}) = \delta_{irr} - \psi_x - \psi_y + \sum_{x\in S, y\notin S} \delta_{g,S}$

(4) $\xi^*(\delta_{h,S}) = \begin{cases} \delta_{h,S} & \text{if } \ g = 2h, \quad n = 0 \\ \delta_{h,S} + \delta_{h-1,S\cup\{x,y\}} & \text{otherwise} \end{cases}$

**Exercise 0.3.** Let

$$at_{h,S} : \overline{\mathrm{M}}_{g-h,n-S\cup\{x\}} \to \overline{\mathrm{M}}_{g,n}$$

be the morphism obtained by attaching a fixed curve of genus $h$ and marking $S \cup \{y\}$ to curves in $\overline{\mathrm{M}}_{g-h,n-S\cup\{x\}}$ by identifying $x$ and $y$. Show that the following relations hold:

(1) $at_{h,S}^*(\kappa) = \kappa$.

(2) $at_{h,S}^*(\phi_i) = \begin{cases} \phi_i & \text{if } \ i \in S \\ 0 & \text{otherwise} \end{cases}$

(3) $at_{h,S}^*(\delta_{irr}) = \delta_{irr}$.

(4) If $S = \{1, \ldots, n\}$, then

$$at_{h,S}^*(\delta_{k,T}) = \begin{cases} \delta_{2h-g,S\cup\{x\}} - \psi_x & \text{if } \ k = h, \#T = n, \text{ or } \ k = g-h, \#T = 0 \\ \delta_{k,T} + \delta_{k+h-g,T\cup\{x\}} & \text{otherwise} \end{cases}$$

(5) If $S \neq \{1,\dots,n\}$, then

$$at^*_{h,S}(\delta_{k,T}) = \begin{cases} -\psi_x & \text{if } \ (k,T)=(h,S) \text{ or } \ (k,T)=(g-h,S^c) \\ \delta_{k,T} & \text{if } \ T\subset S \text{ and } \ (k,T)\neq(h,S) \\ \delta_{k+h-g,(T\setminus S^c)\cup\{x\}} & \text{if } \ S^c\subset T \ \text{ and } \ (k,T)\neq(g-h,S^c) \\ 0 & \text{otherwise} \end{cases}$$

**Exercise 0.4.** Using the previous exercises and our calculations in class determine the divisor class relations between $\kappa$, $\psi$ and $\delta$ classes in $\overline{\mathrm{M}}_{1,n}$ and $\overline{\mathrm{M}}_{2,n}$.