18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 8

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1. Examples

1.1. **Linear systems on $\mathbb{P}^2$.** Let P be a linear system (of conics, cubics, etc.) on $\mathbb{P}^2$ and $\phi : \mathbb{P}^2 \dashrightarrow P^\vee \cong \mathbb{P}^N$ the corresponding rational map. The full linear system of degree k polynomials has dimension $N = \binom{k+2}{2} - 1$. ϕ may have base points: blow them up to get $f = \phi \circ \pi : S = \mathbb{P}^2(r) \to \mathbb{P}^N$ with exceptional divisors corresponding to base points $p_1, \ldots, p_r$ (here, we assume one blowup is sufficient to resolve each point). Let m_i be the minimal multiplicity of the members of the linear system at p_i, d the degree of S. Let ℓ be line in $\mathbb{P}^2$, $L = \pi^*\ell, E_i = \pi^{-1}(p_i)$. We obtain $\hat{I} \subset |dL - \sum m_i E_i|$ a linear system without base points on S. Assume f is an embedding, i.e. it separates points and tangent vectors. Then $S' = f(S)$ is a smooth rational surface in $\mathbb{P}^N$ and $\mathrm{Pic}\,(S')$ has an orthogonal basis consisting of $L = \pi^*\ell$ and the E_i with $L^2 = 1, E_i^2 = -1$. The hyperplane section H of S' is $dL - \sum m_i E_i$ and the degree of S' is $H^2 = d^2 - \sum m_i^2$.

Example. The linear system of all conics on $\mathbb{P}^2$ gives an embedding $j : \mathbb{P}^2 \to \mathbb{P}^5$ with no base points via $[x : y : z] \mapsto [x^2 : y^2 : z^2 : xy : xz : yx]$: the image V has degree 4 and is called the Veronese surface. It contains no lines, but contains a two-dimensional linear system of conics coming from lines on $\mathbb{P}^2$. We can write down equations for V in $\mathbb{P}^5$ as a determinantal variety, with

$$\mathrm{rk} \begin{pmatrix} Z_0 & Z_3 & Z_5 \\ Z_3 & Z_1 & Z_4 \\ Z_5 & Z_4 & Z_2 \end{pmatrix} = 1 \tag{1}$$

i.e. all 2×2 minors vanish, i.e. it is cut out by quadratic relations. Projecting from a generic point of $\mathbb{P}^5$ gives an isomorphism $V \to V' \subset \mathbb{P}^4$ called the *Steiner surface*, while projection from a point of V is a surface $S \subset \mathbb{P}^4$ of degree 3 obtained from a linear system of conics passing through a given point on $\mathbb{P}^2$. This in turn gives an embedding $\mathbb{F}_1 \subset \mathbb{P}^4$, a cubic ruled surface in $\mathbb{P}^4$.

Proposition 1. *The linear system of cubics passing through points $p_1, \ldots, p_r, r \leq 6$ in general position (no 3 on a line, no 6 on a conic) gives an embedding $j : P_r = \mathbb{P}^2(r) \to \mathbb{P}^d, d = 9 - r$. $S_d = j(P_r)$ is a surface of degree d in $\mathbb{P}^d$, called a* del Pezzo surface *of degree d.*

$r = \#E_i$	0	1	2	3	4	5	6
$\#\langle p_i, p_j\rangle$	0	0	1	3	6	10	15
# conics	0	0	0	0	0	1	6
# total lines	0	1	3	6	10	16	27

Proof. To see this, we need to check that the linear system of cubics through $p_1, \dots, p_r$ separates points and tangent vectors on P_r. Then the system is without base points, so by induction the dimension is $9 - r$. We only need to check for $r = 6$: to see that it separates points, take $x, y \in P_6$ with $x \neq y$, and choose $p_i \neq \pi(x), \pi(y)$ s.t. x is not on the proper transform of the conic C_i through the 5 points $p_j, j \neq i$. There is a unique conic D_{ijx} through x and the points p_k for $k \neq i, j$. Then $D_{ijx} \cap D_{ikx} = \{x\}$ for $p_k \neq \{p_i, p_j, \pi(x)\}$. Hence $y \in D_{ijx}$ for at most one value of j, and there is some D_{ijx} s.t. $y \notin D_{ijx}$. Also, if L_{ij} is the proper transform of the line joining p_i, p_j, then $y \in L_{ij}$ for at most one value of j. So there is a cubic $D_{ijx} \cup L_{ij}$ passing through x but not j, and $j : P_6 \to \mathbb{P}^3$ is injective. Separating tangent vectors follows similarly. □

Note. The linear system of cubics passing through $p_1, \dots, p_r$ is the complete anticanonical system $-K$ on P_r (as $K = -3H + E_1 + \cdots + E_r$)

Proposition 2. *S_d contains a finite number of lines, which are the images of the exceptional curves E_i, the strict transforms of the lines $\langle p_i, p_j\rangle$, $i \neq j$, and the strict transforms of the conics through 5 of the $\{p_i\}$.*

Proof. Since $H = -K$, the lines on S are its exceptional curves (want $\ell \cdot H = 1 = -K \cdot \ell$, $2g - 2 = -2 = \ell^2 + K \cdot \ell \implies \ell^2 = -1$). In particular, $j(E_i)$ are lines in S. Let E be a divisor on S not equal to some E_i. Then $E \cdot H = 1, E \cdot E_i = 0$ or 1 implies that $E \equiv mL - \sum m_i E_i$ with $m_i \in \{0, 1\}$ for all i, $E \cdot H = 3m - \sum m_i = 1$. Counting all the solutions of these equations, we get all the numbers above and the classes of the lines in Pic S, so we can compute intersection numbers, etc. □

Note. Classically, a del Pezzo surface is defined to be a surface X of degree d in $\mathbb{P}^d$ s.t. $\omega_X \cong \mathcal{O}_X(-1)$ (i.e. it is embedded by its anticanonical bundle). Every del Pezzo surface is a P_r for some $r = 0, \dots, 6$ or is the 2-uple embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ which is a del Pezzo surface of degree 8 in $\mathbb{P}^8$.

If we have $p_1, \dots, p_r$ a finite set of points in $\mathbb{P}^2$, we can define a notion of general position for these (no 3 collinear, no 6 on a conic, even after a finite set of admissible quadratic transformations). If we blow up 7 points in general position, we get exactly 56 irreducible nonsingular exceptional curves of the first kind. For $r = 8$, we get 240 exceptional curves. The numbers are related to the root latices $A_1, A_2, A_5, D_4, D_5, E_6, E_7, E_8$: the automorphism groups of these graphs coming from exceptional curves are related to the Weyl groups of these groups. If we blow up $r = 9$ points, the surface has infinitely many exceptional curves of the first kind.

Theorem 1. *Any smooth cubic surface in* $\mathbb{P}^3$ *is a del Pezzo surface of degree* 3*, i.e. it is isomorphic to* $\mathbb{P}^2$ *with* 6 *points blown up.*

Theorem 2. *Any smooth complete intersection of* 2 *quadrics in* $\mathbb{P}^4$ *is a del Pezzo surface of degree* 4*.*

See Beauville for proofs.

1.2. **Rational normal scrolls.** A scroll is a ruled surface embedded in $\mathbb{P}^N$ s.t. the fibers of ruling are straight lines. The Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^d$ is called the rational normal curve, and comes from $v_d : \mathbb{P}^1 \to \mathbb{P}^d, [x : y] \mapsto [x^d : x^{d-1}y : \cdots : y^d]$. Its image has degree d, which is the minimal degree for a nondegenerate curve in $\mathbb{P}^d$: if you intersect a curve of degree d with a generic hyperplane, you get e points which span $\mathbb{P}^{d-1}$, so $e \geq d$. A *rational normal scroll* $S_{a,b}$ in $\mathbb{P}^{a+b+1}$: take 2 complementary linear subspaces of dimensions a and b, put a rational normal curve in each, and take the union of all lines joining $v_a(p)$ to $v_b(p)$ for $p \in \mathbb{P}^1$. We can also think of $S_{a,b}$ as the quotient of $(\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\})$ by the action of $\mathbb{F}_m \times \mathbb{F}_m$ $(k^* \times k^*)$ acting as

$$\begin{aligned} (\lambda, 1) &: (t_1, t_2, x_1, x_2) \mapsto (\lambda t_1, \lambda t_2, \lambda^{-a} x_1, \lambda^{-b} x_2) \\ (1, \mu) &: (t_1, t_2, x_1, x_2) \mapsto (t_1, t_2, \mu x_1, \mu x_2) \end{aligned} \tag{2}$$

One can check that the fibers are $\mathbb{P}^1$: we get a geometrically ruled surface over $\mathbb{P}^1$, i.e. a rational surface. In fact, $S_{a,b}$ is isomorphic to $\mathbb{F}_n, n = b - a$, and the special curve G of negative self-intersection $-n$ is the rational normal curve of degree a where $1 \leq a \leq b$. The hyperplane divisor $(\mathcal{O}_{\mathbb{F}_n}(1))$ is $G + bF$ and the degree of $S_{a,b}$ is $a + b$. When $a = 0$, we get a cone over a rational normal curve of degree b, while for $a = b = 1$ we get a smooth quadric in $\mathbb{P}^3$.

Theorem 3. *A surface of degree* $n - 1$ *in* $\mathbb{P}^n$ *is either a rational normal scroll* $S_{a,b}$ *or the Veronese surface of degree* 4 *in* $\mathbb{P}^5$*: this is the minimal possible degree for a nondegenerate surface.*

Theorem 4. *A surface of degree* k *in* $\mathbb{P}^k$ *is either a del Pezzo surface or a Steiner surface.*

Next, we will see where ruled and rational surfaces fit into the classifications of surfaces.

Theorem 5 (Enriques)**.** *An algebraic surface with Kodaira dimension* $\kappa(S) = -\infty$ *is ruled.*

Theorem 6 (Castelnuovo)**.** *An algebraic surface is rational iff* $q = p_2 = 0$*.*

Theorem 7. *An algebraic surface has* $\kappa = -\infty$ *iff* $p_4 = p_6 = 0$