18.727 Topics in Algebraic Geometry: Algebraic Surfaces
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ALGEBRAIC SURFACES, LECTURE 6

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Corollary 1. *Assume that all the closed fibers of the morphism* $\pi : X \to B$ *are isomorphic to* $\mathbb{P}^1$ *(i.e.* π *is smooth and the fibers have arithmetic genus* 0*; or say that* $\pi : X \to B$ *is geometrically ruled). Then there exists a locally free* $\mathcal{O}_B$*-module* E *of rank* 2 *and a* B*-isomorphism* $u : X \to \mathbb{P}_B(E)$*. If* E *and* E' *are two locally free* $\mathcal{O}_B$*-modules of rank* 2 *then* $\exists$ *a* B*-isomorphism* $v : \mathbb{P}(E) \to \mathbb{P}(E')$ *if and only if* $\exists$ *an invertible* $\mathcal{O}_B$*-module* L *s.t.* $E' \cong E \otimes_{\mathcal{O}_B} L$*.*

Proof. The first part follows from the proof of the theorem above. E is constructed as $\mathcal{O}_X(D), D = \sigma(B)$ for a section σ. Geometrically, E is patching together $\pi^{-1}(U) \cong U \times \mathbb{P}^1$, so we get a locally free sheaf of rank 2. For the second part, if $E' \cong E \otimes L$, then $\mathbb{P}(E) \cong \mathbb{P}(E')$ is easy. Conversely, assume that $\exists$ a B-isomorphism $v : \mathbb{P}(E) \to \mathbb{P}(E')$. Then by the proposition below (about $\mathrm{Pic}\,(\mathbb{P}(E))$), $v^*\mathcal{O}_{\mathbb{P}(E')}(1) \cong \pi^*(L) \otimes \mathcal{O}_{\mathbb{P}(E)}(S)$, where $s \in \mathbb{Z}, \pi : \mathbb{P}(E) \to B$ (resp. $\pi' : \mathbb{P}(E') \to B$) are the canonical projections. L is an invertible $\mathcal{O}_B$-module, and

$$\begin{aligned} E' &\cong \pi'_*\mathcal{O}_{\mathbb{P}(E')}(1) \cong \pi_* v^*\mathcal{O}_{\mathbb{P}(E')}(1) \cong \pi_*(\pi^*(L) \otimes \mathcal{O}_{\mathbb{P}(E)}(S)) \\ &\cong L \otimes \pi_*\mathcal{O}_{\mathbb{P}(E)}(S) \cong L \otimes \mathrm{Sym}^s E \end{aligned} \tag{1}$$

Comparing ranks, we see that $s = 1$, so $E' \cong L \times E$. □

Proposition 1. *Let* $f : \mathbb{P}_Z(E) \to Z$ *be a projective bundle with* Z *an arbitrary variety,* E *a locally free* $\mathcal{O}_Z$*-module of finite rank. Then* $\mathrm{Pic}\,(\mathbb{P}(E)) \cong \mathbb{Z}[\mathcal{O}_{\mathbb{P}(E)}(1)] \oplus f^*(\mathrm{Pic}\,(Z))$*.*

Proof. This follows from the base change theorem, e.g. ex II.7.9 in Hartshorne. We will see this explicitly for ruled surfaces later. □

Remark. Another proof of the the second part of the corollary: $\mathbb{P}(E)$ is a $\mathbb{P}^1$-bundle over the base B, and the set of isomorphism classes of such bundles can be identified with the set $H^1(B, G)$, where G is the sheaf of nonabelian groups defined by $G(U) = \mathrm{Aut}_U(U \times \mathbb{P}^1) = \{\text{morphisms of } U \text{ into } \mathrm{PGL}_2(k)\}$ Now, let $G = \mathrm{PGL}_2(\mathcal{O}_B)$: we have

$$1 \to \mathcal{O}_B^* \to \mathrm{GL}_2(\mathcal{O}_B) \to \mathrm{PGL}_2(\mathcal{O}_B) \to 1 \tag{2}$$

giving the associated long exact sequence

$$H^1(B, \mathcal{O}_B^*) \to H^1(B, \mathrm{GL}_2(\mathcal{O}_B)) \to H^1(B, \mathrm{PGL}_2(\mathcal{O}_B)) \to H^1(B, \mathcal{O}_B^*) \tag{3}$$

The first object is $\mathrm{Pic}\,(B)$, the last is 0 since B is a curve, while the second and third are respectively the set of isomorphism classes of rank 2 vector bundles and the set of isomorphism classes of $\mathbb{P}^1$-bundles.

Lemma 1. *Let D be an effective divisor on a surface X, C an irreducible curves s.t. $C^2 \geq 0$. Then $D.C \geq 0$.*

Proof. $D = D' + nC$, where D' does not contain C and $n \geq 0$. Then $D \cdot C = D' \cdot C + nC^2 \geq 0$. □

Lemma 2. *Let p be a surjective morphism from a surface to a smooth curve with connected fibers and $F = \sum n_i C_i$ a reducible fiber. Then $C_i^2 < 0$ for all i.*

Proof. $n_i C_i^2 = C_i(F - \sum_{j \neq i} n_j C_j) = 0 - \sum_{j \neq i} n_j (C_i - C_j)$. $(C_i \cdot C_j) \geq 0$, with at least one being > 0 (because F is connected), so the sum is negative. □

Lemma 3. *Let X be a minimal surface, B a smooth curve, $\pi : X \to B$ a morphism with generic fiber isomorphic to $\mathbb{P}^1$. Then X is geometrically ruled by π.*

Proof. Let F be a fiber of π: then $F^2 = 0 \implies F \cdot K = -2$ by the genus formula. If F is reducible, $F = \sum n_i C_i$: applying the genus formula and the above lemma, we find that $K \cdot C_i \geq -1$, with equality $\Leftrightarrow C_i^2 = -1 (-2 \leq 2g(C_i) - 2 = C_i^2 + C_i \cdot K \leq -1 + K \cdot C_i)$. This would imply that C_i is an exceptional curve, contradicting the minimality of X. So $K \cdot C_i \geq 0 \implies K \cdot F = \sum n_i (K \cdot C_i) \geq 0$, contradicting $K \cdot F = -1$. So F must be irreducible. It cannot be a multiple, since $F = aF' \implies (F')^2 = 0, aF' \cdot K = F \cdot K = -1 \implies a = 2$ and $F' \cdot K = -1$ which is again impossible. F is therefore integral and isomorphic to $\mathbb{P}^1$ (arithmetic genus 0), and π is smooth on $F \implies \pi : X \to B$ is smooth. □

Theorem 1. *Let B be a smooth, irrational (i.e. $g > 0$) curve. The minimal models of $B \times \mathbb{P}^1$ are exactly the geometrically ruled surfaces over B, i.e. the $\mathbb{P}^1$-bundles $\mathbb{P}_B(E)$.*

Proof. Let $\pi : X \to C$ be geometrically ruled. If E is an exceptional curve, then E cannot be a fiber of π since $E^2 = -1$. So $\pi(E) = B$, which is not possible since E is rational and B has higher genus. Thus, X is minimal.

Conversely, suppose X is minimal and $\phi : X \dashrightarrow B \times \mathbb{P}^1$ is birational. Let $q : B \times \mathbb{P}^1 \to B$ be the projection, and consider $q \circ \phi$. There is a diagram factoring this map through a sequence of blowups $X' \xrightarrow{\epsilon_n} \cdots \xrightarrow{\epsilon_1} X$ as a map $f : X' \to B$. Suppose $n > 0$, and let E be the exceptional curve for ϵ_n. Since B is not rational, $f(E)$ must be a single point, so f factors as $f' \circ \epsilon_n$, contradicting the minimality of n. So $n = 0$, and $q \circ \phi$ is a morphism with rational generic fiber. The lemma above shows that X is geometrically ruled by $q \circ \phi$. □

Let $X = \mathbb{P}_B(E)$ be a geometrically ruled surface over B, $\pi : X \to B$ the structure map. The bundle π^*E on X has a natural subbundle N. Over a point $x \in X$, consider the corresponding line $D \subset E_{\pi(X)}$, and let $N_X = D$. The bundle $\mathcal{O}_X(1)$ (the tautological bundle on X) is defined by

$$0 \to N \to \pi^*E \to \mathcal{O}_X(1) \to 0 \tag{4}$$

Let Y be any variety, $f : Y \to B$ a morphism. If there is a morphism $g : Y \to \mathbb{P}(E)$ s.t. degree commutes, then we can associate a line bundle $L = g^*\mathcal{O}_X(1)$ and the surjective morphism $g^*u : g^*\pi^*E = f^*E \to L$. Conversely, given a line bundle L on Y and a surjective morphism $v : f^*E \to L$, we can define a B-morphism $g : Y \to \mathbb{P}(E)$ by associating to $y \in Y$ the line $\operatorname{Ker}(v_y) \subset E_{f(y)}$. These two constructions are inverse to each other, and, in particular, giving a section $\sigma : B \to X = \mathbb{P}(E)$ of π is equivalent to giving a quotient line bundle of $E = \mathrm{id}^*E$.

Proposition 2. *Let $X = \mathbb{P}_B(E)$ be a geometrically ruled surface, and let $\pi : X \to B$ be the structure map. Let n be the class of $\mathcal{O}_X(1)$ in $\operatorname{Pic}(X)$, and let f be the class of the fiber. Then (a) $\operatorname{Pic} X = \pi^*\operatorname{Pic} B \oplus \mathbb{Z}h$ and (b) $\operatorname{Num} X = \mathbb{Z}f + \mathbb{Z}h$.*

Proof. For (a), let h be the class of $\mathcal{O}_X(1)$. It is clear that $h \cdot f = 1$. Now, let $D \in \operatorname{Pic} X$, $n = D \cdot f, D' = D - nh$ so $D' \cdot f = 0$. It is enough to show that D' is the pullback under π^* of a divisor on B. Let $D_n = D' + nF$ for F a fiber, $D_n^2 = D^2$. Also, $D_n \cdot K = D' \cdot K + nF \cdot K = D' \cdot K - 2n$, and $h^0(K - D_n) = 0$ for n sufficiently large. Riemann-Roch for D_n gives $h^0(D_n) \geq \frac{1}{2}D_n(D_n - K) = O(n)$. Thus, $|D_n|$ is nonempty for large enough n. Let $E \in \mathbb{P}^1_n$. Since $E \cdot F = 0$, every component of E is vertical, so it is a fiber of π and thus the inverse image of a divisor on B. This implies that D' is as well, proving our claim. (b) follows from this and the fact that $\operatorname{Num} B = \mathbb{Z}$ generated by the class of a point. □

Lemma 4. *Let E be a locally free sheaf of rank 2 on a curve B. Then there is an exact sequence $0 \to L \to E \to M \to 0$ with L and M line bundles on B.*

Proof. We may twist E by a very ample line bundle H so that $E \otimes H$ is generated by global sections. If we can prove the statement for $E \otimes H$, tensoring by H^{-1} gives the statement for E. So let $s_1, \ldots, s_k$ be global sections which generate E. We claim that there is an element in the span of these sections s.t. $s_b \notin m_b E_b$ for every $b \in B$. Consider the incidence correspondence $\Sigma \subset B \times \mathbb{P}^{k-1} = \{(b, s) | s(b) = 0\}$. Σ is an irreducible variety of dimension $k - 3 + 1 = k - 2$, and thus one cannot cover all of $\mathbb{P}^{k-1}$ by projections of such correspondences. This gives us a sequence $O \to \mathcal{O}_X \to E \to E/s\mathcal{O}_X \to 0$ with $E/s\mathcal{O}_X$ locally free, implying the desired exact sequence. □

Remark. The above generalizes to higher dimensions. Also, the same argument shows that, for every locally free sheaf E of rank $r \geq 2$ on a curve B, there is a sequence $0 \subset E_0 \subset \cdots \subset E_n = E$ of subsheaves s.t. E_i/E_{i-1} are invertible.