18.727 Topics in Algebraic Geometry: Algebraic Surfaces
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ALGEBRAIC SURFACES, LECTURE 4

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We recall the theorem we stated and lemma we proved from last time:

Theorem 1. *Let $f : X \dashrightarrow S$ be a birational morphism of surfaces s.t. f^{-1} is not defined at a point $p \in S$. Then f factors as $f : X \xrightarrow{g} \tilde{S} \xrightarrow{\pi} S$ where g is a birational morphism and π is the blowup of S at p.*

Lemma 1. *Let S be an irreducible surface, possibly singular, and S' a smooth surface with a birational morphism $f : S \to S'$. Suppose f^{-1} is undefined at $p \in S$. Then $f^{-1}(p)$ is a curve on S.*

Lemma 2. *Let $\phi : S \dashrightarrow S'$ be a birational map s.t. ϕ^{-1} is undefined at a point $p \in S'$. Then there is a curve C on S s.t. $\phi(C) = \{p\}$.*

Proof. ϕ corresponds to a morphism $f : U \to S'$ where U is some open set in S. Let $\Gamma \subset U \times S'$ be the graph of f, and let S_1 denote its closure in $S \times S'$. S_1 is irreducible but may be singular.

$$\begin{array}{ccc} S_1 & & \\ {\scriptstyle q}\downarrow & \searrow^{q'} & \\ S & \xrightarrow[\phi]{} & S' \end{array} \tag{1}$$

The projections q, q' are birational morphisms and the diagonal morphism commutes. Since $\phi^{-1}(q)$ is not defined, $(q')^{-1}(p)$ is not defined either, so $\exists C_1 \subset S$ an irreducible curve s.t. $q'(C_1) = \{p\}$. Moreover, $q(C_1) = C$ is a curve in S: if not, since $S_1 \subset S \times S'$, $q(C_1)$ a point $\implies C_1 \subset \{x\} \times S'$ for some $x \in S$; but such a C_1 can only intersect the graph of f in $\{(x, f(x))\}$ so the closure of the graph of f can't contain the curve C_1. By construction, C contracts to $\{p\}$ under ϕ. □

Proof of theorem. Let $g = \pi^{-1} \circ f$ be the rational map in question. We need to show that g is a morphism. Let $s = g^{-1}$, and suppose that g is undefined at a point $q \in X$.

$$\begin{array}{ccc} X & & \\ {\scriptstyle q}\downarrow\uparrow{\scriptstyle s} & \searrow^{f} & \\ \tilde{S} & \xrightarrow[\pi]{} & S \end{array} \tag{2}$$

Applying the second lemma, we obtain a curve $C \subset \tilde{S}$ s.t. $s(C) = \{q\}$. Then $\pi(C) = f(q)$ by composing $s(C) = \{q\}$ with f. So we must have $C = E$, the exceptional divisor for π, and $f(q) = p$. Let $\mathcal{O}_{x,q}$ be the local ring of X at q, and let $\mathfrak{m}_q$ be its maximal ideal. We claim that there is a local coordinate y on S at p s.t. $f^*y \in \mathfrak{m}_q^2$. To see this, let (x,t) be a local system of coordinates at p. If $f^*t \in \mathfrak{m}_q^2$ then we are done. If not, i.e. $f^*t \notin \mathfrak{m}_q^2$, then f^*t vanishes on $f^{-1}(p)$ with multiplicity 1, so it defines a local equation for $f^{-1}(p)$ in $\mathcal{O}_{X,q}$. So $f^*(x) = u \cdot f^*t$ for some $u \in \mathcal{O}_{x,q}$. Let $y = x - u(q)t$; then

$$f^*y = f^*x - u(q)f^*t = uf^*(t) - u(q)f^*(t) = (u - u(q))f^*(t) \in \mathfrak{m}_q^2 \tag{3}$$

Next, let e be any point on E where s is defined. Then we have $s^*f^*y = (f \circ s)^*y = \pi^*y \in \mathfrak{m}_e^2$. This holds for all e outside a finite set. But π^*y is a local coordinate at every point of E except one, by construction, giving the desired contradiction. □

This proves the universal property of blowing up. Here is another:

Proposition 1. *Every morphism from $\tilde{S}$ to a variety X that contracts E to a point must factor through S.*

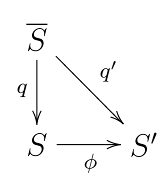
Proof. We can reduce to X affine, then to $X = \mathbb{A}^n$, then to $X = \mathbb{A}^1$. Then f defines a function on $\tilde{S} \setminus E \cong S \setminus \{p\}$ which extends. □

Theorem 2. *Let $f : S \to S_0$ be a birational morphism of surfaces. Then $\exists$ a sequence of blowups $\pi_k : S_k \to S_{k-1}$ ($k = 1, \ldots, n$) and an isomorphism $S \xrightarrow{\sim} S_n$ s.t. $f = \pi_1 \circ \cdots \circ \pi_n \circ u$, i.e. f factors through blowups and an isomorphism.*

Proof. If f^{-1} is a morphism, we're done. Otherwise, $\exists$ a point p of S_0 where f^{-1} is not defined. Then $f = \pi_1 \circ f_1$, where π_1 is the blowup of S_0 at p. If f_1^{-1} is a morphism, we are done, otherwise we keep going. We need to show that this process terminates. Note that the rank of the Neron-Severi group $\operatorname{rk} NS(S_k) = 1 + \operatorname{rk} NS(S_{k-1})$: since $\operatorname{rk}(S)$ is finite, this sequence must terminate. More simply, since f contracts only finitely many curves, it can only factor through finitely many distinct blowups. □

Corollary 1. *Any birational map $\phi : S \dashrightarrow S'$ is dominated by a nonsingular surface $\overline{S}$ with birational morphisms $q, q' : \overline{S} \to S, S'$ which are compositions of blow-up maps, i.e.*

(4)



Proof. First resolve the indeterminacy of ϕ using $\overline{S}$ and then note that q' is a birational morphism, i.e. a composition of blowups by the above. □

1. Minimal Surfaces

We say that a surface S_1 dominates S_2 if there is a birational morphism $S_1 \to S_2$. A surface S is minimal if it is minimal up to isomorphism in its birational equivalence class with respect to this ordering.

Proposition 2. *Every surface dominates a minimal surface.*

Proof. Let S be a surface. If S is not minimal, $\exists$ a birational morphism $S \to S_1$ that is not an isomorphism, so $\operatorname{rk} NS(S) > \operatorname{rk} NS(S_1)$. If S_1 is minimal, we are done: if not, continue in this fashion, which must terminate because $\operatorname{rk} NS(S)$ is finite. $\square$

Note. We say that $E \subset S$ is exceptional if it is the exceptional curve of a blowup $\pi : S \to S'$. Clearly an exceptional curve E is isomorphism to $\mathbb{P}^1$ and satisfies $E^2 = -1$ and $E \cdot K_S = -1$ (since $-2 = 2g - 2 = E \cdot (E + K)$.

Theorem 3 (Castelnuovo)**.** *Let S be a projective surface and $E \subset S$ a curve $\cong \mathbb{P}^1$ with $E^2 = -1$. Then $\exists$ a morphism $S \to S'$ s.t. it is a blowup and E is the exceptional curve (classically called an "exceptional curve of the first kind").*

Proof. We will find S' as the image of a particular morphism from S to a projective space: informally, we need a "nearly ample" divisor which will contract E and nothing else. Let H be very ample on S s.t. $H^1(S, \mathcal{O}_S(H)) = 0$ (take any hyperplane section $\tilde{H}$, then $H = n\tilde{H}$ will have zero higher cohomology by Serre's theorem). Let $k = H \cdot E > 0$, and let $M = H + kE$. Note that $M \cdot E = (H + kE) \cdot E = k + kE^2 = 0$. This M will define the morphism $S \to \mathbb{P}(H^0(S, \mathcal{O}_S(M)))$ (i.e. some $\mathbb{P}^n$). Now, $\mathcal{O}_S(H)|_E \cong \mathcal{O}_E(k)$ since $E \cong \mathbb{P}^1$ and $\deg \mathcal{O}_S(H)|_E = H \cdot E = k$ and on $\mathbb{P}^1$, line bundles are determined by degree. Thus, $\mathcal{O}_S(M)|_E \cong \mathcal{O}_E$.

Now, consider the exact sequence

(5)
$$O \to \mathcal{O}_S(H + (i-1)E) \to \mathcal{O}_S(H + iE) \to \mathcal{O}_E(k - i) \cong \mathcal{O}_S(H + iE)|_E \to 0$$

for $1 \le i \le k + 1$. We know that $H^1(E, \mathcal{O}_E(k - i)) = 0$, so we get

(6)
$$\begin{aligned} 0 \to & H^0(S, \mathcal{O}_S(H + (i-1)E)) \to H^0(S, \mathcal{O}_S(H + iE)) \to H^0(E, \mathcal{O}_E(k - i)) \to \\ & H^1(S, \mathcal{O}_S(H + (i-1)E)) \to H^1(S, \mathcal{O}_S(H + iE)) \to 0 \end{aligned}$$

Thus, the latter map is surjective for $i = 1, \ldots, k+1$: for $i = 0$, $H^1(S, \mathcal{O}_S(H)) = 0$ so all those $H^1(S, \mathcal{O}_S(H + iE))$ are zero.

Next, we claim that M is generated by global sections. Since M is locally free of rank 1: this just means that at any given point of S, not all elements of $H^0(S, \mathcal{O}_S(M))$ vanish, i.e. this linear system has no base point. Since H is very ample, $M = H + kE$ certainly is generated by global sections away from E. On the other hand, $H^0(S, M) \to H^0(S, M|_E)$ is surjective because $H^1(S, H + (k-1)E) = 0$. So it is enough to show that $M|_E$ is generated by global sections on E. Now, $M|_E \cong \mathcal{O}_E(k-k) = \mathcal{O}_E$ is generated by the global section 1. Therefore, lifting it to a section of $H^0(S, M)$ and using Nakayama's lemma, we see that M is generated by global sections at every point of E as well.

So M defines a morphism $S \xrightarrow{f'} \mathbb{P}^n$ for some N. Let S' be the image. Since $(f')^*\mathcal{O}(1) = M$ and $\deg M|_E$ is 0, we see that f' maps E to a point p'. On the other hand, since H is very ample, $H + kE$ separates points and tangent vectors away from E as well as separates points of E from points outside E. So f' is an isomorphism from $S - E \to S' \smallsetminus \{p'\}$.

Let S_0 be the normalization of S'. Since S is nonsingular, hence normal, the map f' factors through S_0 to give a map $f : S \to S_0$. E irreducible $\implies f(E)$ is a point p (the preimage of p' is a finite number of points). We still have $f : S \smallsetminus E \cong S_0 \smallsetminus \{p\}$. We are left to show that S_0 is nonsingular. We show this using Grothendieck's theorem on formal functions: if $f : X \to Y$ is a proper map, $\mathcal{F}$ a coherent sheaf on X, then

$$R^i f_*(\mathcal{F})_y^\wedge \xrightarrow{\sim} \varprojlim H^i(X_n, \mathcal{F}_n) \tag{7}$$

where $X_n = X \times_y \operatorname{Spec} \mathcal{O}_Y/\mathfrak{m}_y^n$ is the thickened scheme-theoretic preimage of y. We'll apply it with $i = 0, \mathcal{F} = \mathcal{O}_S, f : S \to S_0$. $f_*\mathcal{O}_S = \mathcal{O}_{S_0}$ since S_0 is normal. Moreover, $\hat{\mathcal{O}}_p = \varprojlim H^0(E_n, \mathcal{O}_{E_n})$. Now, it is enough to show that $\hat{\mathcal{O}}_p$ is 2-dimensional $\cong k[[x, y]]$. Let's show for every n,

$$H^0(E_n, \mathcal{O}_{E_n}) \cong k[[x, y]]/(x, y)^n \cong k[[x, y]]/(x, y)^n \tag{8}$$

For $n = 1, H^0(E, \mathcal{O}_E) = k$. For $n > 1$, we have

$$0 \to \mathcal{I}_E^n/\mathcal{I}_E^{n+1} \to \mathcal{O}_{E_{n+1}} \to \mathcal{O}_{E_n} \to 0 \tag{9}$$

where $E \cong \mathbb{P}^1 \implies \mathcal{I}_E/\mathcal{I}_E^2 \equiv \mathcal{O}_{\mathbb{P}^1}(1), \mathcal{I}_E^n/\mathcal{I}_E^{n+1} \cong \mathcal{O}_{\mathbb{P}^1}(n)$. Using the LES, we obtain

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^1}(n)) \to H^0(\mathcal{O}_{E_{n+1}}) \to H^0(\mathcal{O}_{E_n}) \to 0 \tag{10}$$

When $n = 1$, $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ is a 2-dimensional vector space. Taking a basis x, y, $H^0(\mathcal{O}_{E_2})$ (which contains k) is seen to be $k[x, y]/(x, y)^2 = k \oplus kx \oplus ky$. Now inducting, we find that $H^0(\mathcal{O}_{E_n})$ is isomorphic to $k[x, y]/(x, y)^n$. Lift elements x, y to $H^0(\mathcal{O}_{E_{n+1}})$, we find that $H^0(\mathcal{O}_{\mathbb{P}^1}(n)$ is a vector space with basis $x^n, x^{n-1}y, \ldots, y^n$ (contained in the symmetric power of (x, y)). So we get $H^0(\mathcal{O}_{E_{n+1}}) \cong k[x, y]/(x, y)^{n+1}$. The truncations are compatible, so $\hat{\mathcal{O}}_p \cong k[[x, y]] \implies p$ is nonsingular. $\square$