18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 3

LECTURES: ABHINAV KUMAR

1. Birational maps continued

Recall that the blowup of X at p is locally given by choosing $x, y \in \mathfrak{m}_p$, letting U be a sufficiently small Zariski neighborhood of p (on which x and y are regular functions that vanish simultaneously only at the point p), and defining $\tilde{U}$ by $xY - yX = 0$ in $U \times \mathbb{P}^1$. If, for some $q \in U, q \neq p, x(q) \neq 0$, then $Y = \frac{y}{x}X$ and similarly if $y \notin \mathfrak{m}_q$. So we obtain an isomorphism $\tilde{U} \to U$ at q, and $\tilde{U} \xrightarrow{\pi} U$ fails to be an isomorphism only at p, where $\pi^{-1}(p) = \mathbb{P}^1$ is the exceptional divisor E. Note that the blowup $\tilde{X}$ does not depend on the choice of x, y.

Proposition 1. *If C is a curve passing through $p \in X$ with multiplicity m,*

$$\pi^* C = \tilde{C} + mE \tag{1}$$

Proof. Choose local coordinates x, y in a neighborhood of p s.t. $y = 0$ is not tangent to any branch of C at p. Then in $\hat{O}_{x,p}$ we can expand the equation of C in a power series

$$f = f_m(x, y) + f_{m+1}(x, y) + \cdots \tag{2}$$

with $f_m(1, 0) \neq 0$ and each f_k a homogeneous polynomial of degree k. In a neighborhood of $(p, \infty = [1 : 0]) \in \tilde{U} \subset U \times \mathbb{P}^1$, we have local coordinates x and $t = \frac{y}{x}$ and $\pi^* f = f(x, tx) = x^m f_m(1, t) + x^{m+1} f_{m+1}(1, t) + \cdots)$, giving the desired formula. □

Theorem 1. *We have maps $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \tilde{X}$ and $\mathbb{Z} \to \operatorname{Pic} \tilde{X}, 1 \mapsto E$ giving rise to an isomorphism $\operatorname{Pic} \tilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$. If $C, D \in \operatorname{Pic} X$, $(\pi^* C) \cdot (\pi^* D) = C \cdot D$, while $(\pi^* C) \cdot E = 0$ and $E \cdot E = -1$. We further have that $K_{\tilde{X}} = \pi^* K_X + E$, so $K_{\tilde{X}}^2 = (\pi^* K_X)^2 - 1$.*

Proof. Note that $\operatorname{Pic} X \cong \operatorname{Pic}(X \setminus \{p\}) \cong \operatorname{Pic}(\tilde{X} \setminus E)$ and we have $\mathbb{Z} \to \operatorname{Pic} \tilde{X} \to \operatorname{Pic}(\tilde{X} \setminus E) \to 0$. The first map is injective because $E^2 = -1$, and π^* splits the sequence to give the desired isomorphism. For the intersection formulae, move C, D so that they meet transversely and do not pass through p. Because π^* is an isomorphism $\tilde{X} \setminus E \to X \setminus \{p\}$, we get an equality of intersection numbers as desired. Moreover, since C (possibly after moving) does not pass through p, $(\pi^* C)\cdot E = 0$. Next, taking a curve passing through p with multiplicity 1, its strict transform meets E transversely at one point which corresponds to the tangent direction of $p \in C$, i.e. $\tilde{C}\cdot E = 1$ and $\tilde{C} = \pi^*C - E$. Since $(\pi^*C)\cdot E) = 0$, we get $1 = \tilde{C}\cdot E = (\pi^*C - E)\cdot E = -E^2$ as desired. Finally, to show the desired result about canonical divisors, we use the adjunction formula $-2 = 2(0) - 2 = E(E + K_{\tilde{X}}) = -1 + E\cdot K_{\tilde{X}} \implies E\cdot K_{\tilde{X}} = -1$. By the previous proposition, $K_{\tilde{X}} = \pi^*K_X + nE \implies n = 1$ (by taking intersection with E).

Note that we can see this latter fact more directly. Letting $\omega = dx\wedge dy$ be the top differential in local coordinates at p, then $\pi^*\omega = dx \wedge d(xt) = x dx\wedge dt \implies \pi^*K_X + E = K_{\tilde{X}}$. □

1.1. Invariants of Blowing Up.

Theorem 2. *$\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_{\tilde{X}} = 0$ for $i > 0$, so the two structure sheaves have the same cohomology.*

Proof. π is an isomorphism away from E, $\pi : \tilde{X} \setminus E \to X \setminus \{p\}$, so it is clear that $\mathcal{O}_X \to \pi_*\mathcal{O}_{\tilde{X}}$ is an isomorphism except possibly at p, and $R^i\pi_*\mathcal{O}_{\tilde{X}}$ can only be supported at p. By the theorem on formal functions, the completion at p of this sheaf is $\widehat{R^i\pi_*\mathcal{O}_{\tilde{X}}} = \varprojlim H^i(E_n, \mathcal{O}_{E_n})$, where E_n is the closed subscheme defined on $\tilde{X}$ by $\mathcal{I}^n$, $\mathcal{I}$ the ideal sheaf of E. We obtain an exact sequence $0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{E_{n+1}} \to \mathcal{O}_{E_n} \to 0$ with $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_E(1) \implies \mathcal{I}^n/\mathcal{I}^{n+1} \cong \mathcal{O}_E(n)$. Since $E \cong \mathbb{P}^1$, we have $H^i(E, \mathcal{O}_E(n)) = 0$ for $n, i > 0$. Using the long exact sequence in cohomology, we find that $H^i(E_n, \mathcal{O}_{E_n}) = 0$ for all $i > 0, n \geq 1$, so the above inverse limit vanishes. $R^i\pi_*\mathcal{O}_{\tilde{X}}$ is concentrated at p and thus equals its own completion, giving the desired vanishing of higher direct image sheaves. Also, $\pi_*\mathcal{O}_{\tilde{X}} = \pi_*\mathcal{O}_X$ follows from the fact that X is normal and π is birational (trivial case of Zariski's main theorem). The final statement follows from the spectral sequence associated to H^i and $R^i\pi_*$. □

This implies that the irregularity $q_X = h^1(X, \mathcal{O}_X) = q_{\tilde{X}}$ and geometric genus $p_g(X) = h^2(X, \mathcal{O}_X) = p_g(\tilde{X})$ are invariant under blowup.

2. Rational maps

Let X, Y be varieties, X irreducible.

Definition 1. *A rational map $X \dashrightarrow Y$ is a morphism ϕ from an open subset U of X to Y. Note that if two morphisms $U_1, U_2 \to Y$ agree on some $V \subset U \cap U_2$, they agree on $U_1 \cap U_2$, and thus each rational map has a unique maximal domain U. We say that ϕ is defined at $x \in X$ if $x \in U$.*

Proposition 2. *If X is nonsingular, Y projective, then $X \setminus U$ has codimension 2 or larger.*

Proof. If ϕ is not defined on some irreducible curve C, then $\mathcal{O}_{C,X}$ gives us a valuation $v_C : k(X) \to \mathbb{Z}$. Let ϕ be given by $(f_0 : \cdots : f_n)$ with $f_i \in K(X)$ s.t. at least one f_j has a pole along C. Take the f_i s.t. $v(f_i)$ is the smallest, and divide by it. Then ϕ is defined on the generic point of C, a contradiction. □

In particular, if X is a smooth surface and Y is projective, a rational map is defined on all but finitely many points F (those lying on the set of zeroes and poles of ϕ). If C is an irreducible curve on X, ϕ is defined on $C \setminus (C \cap F)$, and we can set $\phi(C) = \overline{\phi(C \setminus (C \cap F))}$ (and similarly $\phi(X) = \overline{\phi(X \setminus F)}$). Restriction gives us an isomorphism between $\mathrm{Pic}\,(X)$ and $\mathrm{Pic}\,(X \setminus F)$, so we can talk about the inverse image of a divisor D (or line bundle, or linear system) under ϕ.

3. Linear Systems

For a divisor D, $|D|$ is the set of effective divisors linearly equivalent to D, i.e. $\mathbb{P}(H^0(X, \mathcal{O}_X(D))^\vee)$. A hyperplane in this projective space pulls back to give a divisor equivalent to D. For $f \in H^0(\mathcal{O}_X(D))$, let D' be the divisor of zeroes of f. A complete linear system is such a space $\mathbb{P}(H^0(X, \mathcal{O}_X(D))^\vee)$, while a linear system P is simply a linear subspace of such a system. One dimensional linear systems are called pencils. A component C of P is called fixed if every divisor of P contains C, i.e. all the elements of the corresponding subspace of $H^0(X, \mathcal{O}_X(D))^\vee$ vanish along C. The fixed part of P is the biggest divisor F which is contained in every element of P, so that $|D - F|$ for $D \in P$ has no fixed part. A point p of X is a base point of P if every divisor of P contains p. If p has no fixed part, then it has finitely many base points (at most (D^2)).

4. Properties of Birational Maps between Surfaces

(1) Elimination of indeterminacy
(2) Universal property of blowing up
(3) Factoring birational morphisms
(4) Minimal surfaces
(5) Castelnuovo's contraction theorem

Theorem 3. *Let $\phi : S \dashrightarrow X$ be a rational map from a surface to a projective variety. Then $\exists$ a surface S', a morphism $\eta : S' \to S$ which is the composition of a finite number of blowups, and a morphism $f : S' \to X$ s.t. f and $\phi \circ \eta$ coincide.*

Proof. We may as well assume that $X = \mathbb{P}^n$ and $\phi(S)$ is not contained in any hyperplane of $\mathbb{P}^n$. So ϕ corresponds to a linear system $P \subset |D|$ of dimension m with no fixed component. If P has no base points, ϕ is an isomorphism and we are done. Otherwise, let p be such a base point, and consider the corresponding blowup $\pi : X_1 \to S$. The exceptional curve E occurs in the fixed part of $\pi^* P \subset |\pi^* D|$ with some multiplicity $k \geq 1$ (i.e. the smallest multiplicity of curves in P passing through p). Then $P_1 \subset |\pi^* D - kE|$ obtained by subtracting kE from elements of $\pi^* P$ has no fixed component, and defines a rational map $\phi_1 : X_1 \dashrightarrow \mathbb{P}^m$ which coincides with $\phi \circ \pi$. If ϕ_1 is a morphism, we are done; otherwise, repeat the process. We obtain a sequence of divisors $D_n = \pi_n^* D_{n-1} - k_n E_n \implies 0 \leq D_n^2 = D_{n-1}^2 - k^2 < D_{n-1}^2$, which must terminate. □

Theorem 4. *Let $f : X \dashrightarrow S$ be a birational morphism of surfaces s.t. f^{-1} is not defined at a point $p \in S$. Then f factors as $f : X \xrightarrow{g} \tilde{S} \xrightarrow{\pi} S$ where g is a birational morphism and π is the blowup of S at p.*

Lemma 1. *Let S be an irreducible surface, possibly singular, and S' a smooth surface with a birational morphism $f : S \to S'$. Suppose f^{-1} is undefined at $p \in S$. Then $f^{-1}(p)$ is a curve on S.*

Proof. We may assume that S is affine, with $f^{-1}(p)$ nonempty, so there is an embedding $j : S \to \mathbb{A}^n$. Now, $j \circ f^{-1} : S \dashrightarrow \mathbb{A}^n$ is given by rational functions $g_1, g_2, \ldots, g_n$ and at least one of them is undefined at p, say $g_1 \notin \mathcal{O}_{S',p}$. Let $g_1 = \frac{u}{v}$, where $u, v \in \mathcal{O}_{S',p}$ are coprime and $v(p) = 0$. Let D be defined on S by $f^* v = 0$. On S we have $f^* u = (f^* v) x_1$ (where x_1 is the first coordinate function on $S \subset \mathbb{A}^n$) (because it is true under $(f^{-1})^*$): $(f^{-1})^* f^* u = (f^{-1})^* f^* v \cdot (f^{-1})^* x_1$, $k(S) = k(S')$. So $f^* u = f^* v = 0$ on D, and $D = f^{-1}(Z)$ where Z is the subset of S' defined by $u = v = 0$. This is a finite set since u, v are coprime. Shrinking S' if necessary, we can assume $Z = \{p\}$, and $D = f^{-1}(p)$ as desired. □