18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 20

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Last time we stated the following theorem:

Theorem 1. *Let X be a minimal surface with $K^2 = 0$, $K \cdot C \geq 0$ for all curves C on X. Then either $2K \sim 0$ or X has an icoct (indecomposable curve of canonical type).*

Proof. First, assume $|2K| \neq \varnothing$. Let $D \in |2K|$: then either $D = 0$, in which case $2K \sim 0$ and we're done, or else $D = \sum n_i E_i > 0$. Then $D \cdot K = 2K^2 = 0$. So $\sum n_i (K \cdot E_i) = 0$. But $K \cdot E_i \geq 0$ for all i by assumption. This forces $K \cdot E_i = 0$ for all i, so $D \cdot E_i = 0$ for all i as well. Thus, D is of canonical type. We get an icoct by decomposing D.

On the other hand, if $|2K| = \varnothing, K^2 = 0$, so RR gives $h^0(2K) + h^0(-K) \geq \chi(\mathcal{O}_X)$. By assumption, $p_2 = h^0(2K) = 0$, so $p_g = 0$ as well, implying that $\chi(\mathcal{O}_X) = 1 - q \implies h^0(-K) \geq 1 - q$.

If $q = 0$ then $H^0(-K) \neq 0$. Letting $D \in |-K|$, if $D = 0$ then $K \sim 0 \implies 2K \sim 0$, a contradiction. If $D > 0$, then for H ample, $D \cdot H \geq 0 \implies K \cdot H < 0$ contradicting our hypothesis. So assume $q \geq 1, p_g = 0$. Noether's formula gives $10 - 8q = b_2 \implies q \leq 1 \implies q = 1$. Let $f : X \to B = \mathrm{Alb}\,(X)$ be the Albanese map, which in this case must be a surjective map onto an elliptic curve. Let $F_b = f^{-1}(b)$ be the fiber over $b \in B$. If $p_a(F_b) = 0$, then $F_b^2 = 0$ gives $F_b \cdot K = -2$ by the genus formula, a contradiction.

If $p_a(F_b) = 1$, F_b is an icoct and we are done. So assume $p_a(F_b) \geq 2$. The genus formula gives $K \cdot F_b = 2p_a(F_b) - 2 \geq 2$. For any closed point $a \in B \setminus \{b\}$, let F_a be the fiber over a. Then we have a short exact sequence

$$0 \to \mathcal{O}_X(2K + F_a - F_b) \to \mathcal{O}_X(2K + F_a) \to \mathcal{O}_{F_b}(2K|_{F_b}) \to 0 \tag{1}$$

Note that $\mathcal{O}_X(F_a) \otimes \mathcal{O}_{F_b} = \mathcal{O}_{F_b}$. Now, let's check $|F_b - F_a - K| = \varnothing$. Otherwise, we have $D \sim F_b - F_a - K \geq 0$. Then $K \sim F_b - F_a - D$ and $K \cdot F_b = F_b^2 - F_a \cdot F_b - D \cdot F_b = -D \cdot F_b \leq 0$ (because $D \cdot F_b = D \cdot F_c \geq 0$, since we can move the fiber) contradicting $K \cdot F_b \geq 2$. Therefore $H^2(2K + F_a - F_b) = 0$ by Serre duality, and Riemann-Roch gives $\chi(2K + F_a - F_b) = \chi(\mathcal{O}_X) = 1 - q + p_g = 0$. So either $|2K + F_a - F_b| \neq \varnothing$, or $H^i(\mathcal{O}_X(2K + F_a - F_b)) = 0$ for $i = 0, 1, 2$. In the first case, we get the desired icoct by taking an effective divisor in $|2K + F_a - F_b|$. So assume the latter case for all $a \in B \setminus \{b\}$. Then $H^0(\mathcal{O}_X(2K + F_a)) \to$

$H^0(\mathcal{O}_{F_b}(2K|_{F_b}))$ is an isomorphism. Fix a nonzero section $s \in H^0(\mathcal{O}_{F_b}(2K|_{F_b}))$, which exists because F_b has genus ≥ 2 and $\deg(2K|_{F_b}) = 2(K \cdot F_b) = 2(2p_a(F_b) - 2)$, so Riemann-Roch on F_b gives a global section.

Let $\Delta = \operatorname{div}_{F_b}(s)$. For every $a \in B \setminus \{b\}$, lift s uniquely to a section $s_a \in H^0(\mathcal{O}_X(2K + F_a))$. Let $D_a = \operatorname{div}_X(s_a)$. This is an algebraic family of effective divisors $\{D_a\}_{a \in B \setminus \{b\}}$ s.t. $D_a|_{F_b} = \Delta$ for all a. We cannot have $D_a \sim D'_a$ for $a \neq a'$ (else $F_a \sim F'_a \implies a \sim a'$ on the elliptic curve B, which is impossible). So in particular, $D_a \neq D_{a'}$, and X is the closure of $\bigcup_{a \neq b} D_a$. Letting D_b be the specialization of D_a as $a \to b$, we find that it must have F_b among its components, since $D_a|_{F_b} = \Delta$ is supported on a fixed finite set of points on F_b. So $D_b = F_b + D'_b$ for some $D'_b \geq 0$. Since $D_b \in |2K + F_b|$, we get $D'_b \in |2K|$, contradicting $|2K| \neq 0$. □

Corollary 1. *If X is a minimal surface with $K^2 = 0, K \cdot C \geq 0$ for all curves C on X. Then either $2K \sim 0$ or $\exists$ an elliptic/quasi-elliptic fibration $f : X \to B$.*

1. More on elliptic/quasi-elliptic fibrations

Let $f : X \to B$ be an elliptic/quasi-elliptic fibration. By definition, $k(B)$ is algebraically closed in $k(S)$, so all but finitely many fibers are geometrically integral, and there are a finite number of points $b_1, \ldots, b_r \in B$ s.t. $F_b = f^{-1}(b)$ is an icoct for $b \in B \setminus \{b_1, \ldots, b_r\}$. Furthermore, $F_{b_i} = f^{-1}(b_i) = m_i P_i$, with P_i an icoct (since $f_*\mathcal{O}_X = \mathcal{O}_B$, by Stein factorization all the fibers are connected).

The ones for which $m_i \geq 2$ are called multiple fibers of the fibration.

Now, $R^1 f_* \mathcal{O}_X$ is a coherent $\mathcal{O}_B$-module, and $R^1 f_* \mathcal{O}_X \otimes k(b) = H^1(F_b, \mathcal{O}_b)$ for all $b \in B$ (by the base change theorem). It is clear that, for $b \in B \setminus \{b_1, \ldots, b_r\}$, $h^1(F_b, \mathcal{O}_{F_b}) = 1$ (since $H^1(F_b, \mathcal{O}_{F_b}) \cong H^0(F_b, \omega_{F_b})^\vee$, where ω_{F_b} is the dualizing sheaf of F_b, and since $\omega_b \cong \mathcal{O}_{F_b}$, as F_b is an icoct and $h^0(F_b, \mathcal{O}_{F_b}) = 1$). Since B is a curve, $R^1 f_* \mathcal{O}_X = L \oplus T$ for L locally free of rank 1 (invertible) and T torsion (supported at finitely many points). Also $\operatorname{Supp} T \subset \{b_1, \ldots, b_r\}$, and T is an $\mathcal{O}_B$-module of finite length. Now Riemann-Roch gives $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(-F_b))$ for any b (since $F_b \cdot F_b = 0$ and $K \cdot F_b = 0$ from the genus formula. Thus, $\chi(\mathcal{O}_{F_b}) = 0$ for any b, using the short exact sequence $0 \to \mathcal{O}_X(-F_b) \to \mathcal{O}_X \to \mathcal{O}_{F_b} \to 0$, and $h^0(\mathcal{O}_{F_b}) = h^1(\mathcal{O}_{F_b})$, and $b \in \operatorname{Supp} T \Leftrightarrow h^1(\mathcal{O}_{F_b}) \geq 2 \Leftrightarrow h^0(\mathcal{O}_{F_b}) \geq 2$. These fibers F_b are called the *exceptional* or *wild* fibers (purely a characteristic p phenomenon by a theorem of Raynaud).

Theorem 2. *With the above notation, $\omega_X \cong f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X(\sum a_i P_i)$ where $F_{b_i} = m_i P_i$ for $i = 1, \ldots, r$ are all the multiple fibers of f and $0 \leq a_i < m_i$ ($a_i = m_i - 1$ unless F_{b_i} is exceptional) and $\deg(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + \ell(T)$, where $\ell(T)$ is the length of T as an $\mathcal{O}_B$-module.*

Proof. First, let's see that K is vertical. We show this by finding an effective divisor D linearly equivalent to $K + \sum F_{a_i}$ for some finite set of points $\{a_i\} \subset B$.

Assume this for the moment. Then $D \cdot F_b = 0$ for any closed point $b \in B$ since $F_{a_i} \cdot F_b = 0$ and $K \cdot F_b = 0$ (F_b is of canonical type). So it forces all components of D to be contained in the fibers of f, implying that K has the same property.

We can write $K \sim \sum \ell_j F_{y_j} + D$ for some effective D not containing any fibers. Letting $D_1, \ldots, D_s$ be the connected components of D, we have that each D_i is supported (and contained in) some fiber F_{z_i}. Thus, $D_i^2 \leq 0$. If $D_i^2 < 0$ for some i, then $D_i \cdot E < 0$ for some component E of D_i. Then E is a component of the fiber F_{z_i} which must be reducible, implying that $E^2 < 0$. Also, $K \cdot E = \sum \ell_j F_{y_j} \cdot E + D \cdot E = D \cdot E = \sum D_i \cdot E = D_i \cdot E < 0$ and E is an exceptional curve, contradicting minimality. So $D_i^2 = 0$ for all i, and D_i is a rational multiple of the fiber F_{z_i}, implying that $\omega_X \cong f^*(M) \otimes \mathcal{O}_X(\sum a_i P_i)$ for $0 = a_i \leq m_i$, $a_i \in \mathbb{Z}$.

Now we demonstrate the first step, i.e. getting $K + \sum F_{a_i}$ equivalent to an effective divisor. If F_b is not a multiple fiber, then $\omega_{F_b} \cong \mathcal{O}_{F_b}$ (F_b is an icoct), which gives via adjunction $\omega_X \otimes \mathcal{O}_X(F_b) \otimes \mathcal{O}_{F_b} \cong \mathcal{O}_{F_b}$. Also, $\mathcal{O}_X(F_b) \otimes \mathcal{O}_{F_b}$ is trivial (since it has degree 0 along the components and has a global section). So $\omega_X \otimes \mathcal{O}_{F_b} \cong \mathcal{O}_{F_b}$ as well. So take $a_1, \ldots, a_m$ to be m general points of B (s.t. F_{a_i} are not multiple fibers). Then we have $0 \to \omega_X \to \omega_X \otimes \mathcal{O}_X(\sum F_{a_i}) \to \bigoplus \mathcal{O}_{F_{a_i}} \to 0$ (for one a_i, tensoring $0 \to \mathcal{O}_X \to \mathcal{O}_X(F_a) \to \mathcal{O}_{F_a} \to 0$ by ω_X, and using the Chinese Remainder Theorem for more a_i). We get a cohomology exact sequence

$$0 \to H^0(\omega_X) \to H^0(\omega_X \otimes \mathcal{O}_X(\sum F_{a_i})) \to \bigoplus H^0(F_{a_i}) \to H^1(\omega_X) \to \cdots \tag{2}$$

Now $h^1(\omega_X)$ is constant and $\oplus H^0(F_{a_i})$ had dimension m, so for large enough m, we find that $|K + \sum F_{a_i}|$ is not empty.

Getting back to $\omega_X \cong f^*(M) \otimes \mathcal{O}_X(\sum a_i P_i)$, pushing forward by f_* gives

$$f_*(\omega_X) = M \otimes f_*\mathcal{O}_X(\sum a_i P_i) \tag{3}$$

by the projection formula. Now, we claim that $f_*(\mathcal{O}_X(\sum a_i P_i)) \cong \mathcal{O}_B$. We have $\mathcal{O}_X \subset \mathcal{O}_X(\sum a_i P_i) \subset \mathcal{O}_X(\sum (m_i - 1)P_i)$. So it is enough to show that $f_*(\mathcal{O}_X(\sum (m_i - 1)P_i)) \cong \mathcal{O}_B$. This is local on B, so it is enough to show $f_*(\mathcal{O}_X((m_i - 1)P_i)) \cong \mathcal{O}_B$ for a single i. This is isomorphic to $f_*\mathcal{O}_X(m_i P_i) \otimes \mathcal{O}_X(-P_i) \cong \mathcal{O}_B(b_i) \otimes f_*\mathcal{O}_X(-P_i)$, since $f^*(\mathcal{O}_B(b_i)) = \mathcal{O}_X(m_i P_i)$ using the projection formula. It is enough to show that $f_*\mathcal{O}_X(-P_i) \cong \mathcal{O}_B(-b_i)$. We have $f_*\mathcal{O}_X(-mP_i) \cong \mathcal{O}_B(-b_i) \subset f_*\mathcal{O}_X(-P_i) \subset \mathcal{O}_B$ and $\mathcal{O}_B/\mathcal{O}_B(-b_i)$ has length 1. Thus, it is enough to show $f_*(u) : f_*(\mathcal{O}_X(-P_i)) \to f_*\mathcal{O}_X \cong \mathcal{O}_B$ is not an isomorphism (where $u : \mathcal{O}_X(-P_i) \to \mathcal{O}_X$).

We have the diagram

$$
\begin{array}{ccc}
f_*(\mathcal{O}_X(-P_i)^{\otimes m_i}) & \xrightarrow{\sim} & f_*(\mathcal{O}_X(-m_iP_i)) \cong \mathcal{O}_B(-b_i) \\
\downarrow{\scriptstyle (f_*u)^{\otimes m_i}} & & \downarrow{\scriptstyle f_*(u^{\otimes m_i})} \\
(f_*\mathcal{O}_X)^{\otimes m_i} & \xrightarrow{\sim} & f_*(\mathcal{O}_X^{\otimes m_i}) \cong \mathcal{O}_B
\end{array} \tag{4}
$$

If f_*u is an isomorphism, then the left arrow $(f_*u)^{\otimes m_i}$ is an isomorphism, implying that the right one is as well, which is a contradiction since $\mathcal{O}_B(-b_i)$ is strictly a subsheaf of $\mathcal{O}_B$ with nonzero quotient. So $f_*(u)$ is not an isomorphism, and $f_*(\omega_X) = M$.

By Grothendieck's relative duality theorem, noting that the dualizing complex of $f : X \to B$ is $\omega_{X/B} \cong \omega_X \otimes f^*(\omega_B^{-1})$, we have, for L' an invertible $\mathcal{O}_X$-module, that $f_*(L')$ and $R^1f_*(L'^{-1} \otimes \omega_{X/B})$ are dual w.r.t. $\mathcal{O}_B$. So,

(5)

$$f_*(L') \cong \operatorname{Hom}_{\mathcal{O}_B}(R^1(f_*(L'^{-1} \otimes \omega_{X/B}), \mathcal{O}_B) \cong \operatorname{Hom}_{\mathcal{O}_B}(R^1f_*(L'^{-1} \otimes \omega_X), \omega_B)$$

using the projection formula. Applying this to $L' = \omega_X$, we get

$$
\begin{aligned}
M = f_*\omega_X &\cong \operatorname{Hom}_{\mathcal{O}_B}(R^1f_*\mathcal{O}_X, \omega_B) \cong \operatorname{Hom}_{\mathcal{O}_B}(L \oplus T, \omega_B) \\
&\cong \operatorname{Hom}_{\mathcal{O}_B}(L, \omega_B) = L^{-1} \otimes \omega_B
\end{aligned} \tag{6}
$$

so $\omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X(\sum a_iP_i)$, $0 \le a_i < m_i$. □