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18.727 Topics in Algebraic Geometry: Algebraic Surfaces
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ALGEBRAIC SURFACES, LECTURE 19

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Corollary 1. *If D is an indecomposable curve of canonical type (icoct), then $\omega_D \cong \mathcal{O}_D$, where ω_D is the dualizing sheaf of D.*

Proof. By Serre duality, $h^1(\omega_D) = h^0(\mathcal{O}_D) = 0$. We have the short exact sequence

$$0 \to \mathcal{O}_X(K) \to \mathcal{O}_X(K+D) \to \omega_D \to 0 \tag{1}$$

so $\chi(\omega_D) = \chi(\mathcal{O}_X(K+D)) - \chi(\mathcal{O}_X(K)) = \frac{1}{2}((K+D)\cdot D) = 0$ by Riemann-Roch (using $D^2 = 0$ and $D \cdot K = 0$). Thus, $h^0(\omega_D) = 1$. Since ω_D has degree 0 along the E_i,

$$\deg_{E_i}(\mathcal{O}_D \otimes \mathcal{O}_X(K+D) \otimes \mathcal{O}_{E_i}) = (K+D)\cdot E_i = 0 \tag{2}$$

It follows from the proposition last time that $\omega_D \cong \mathcal{O}_D$. □

Corollary 2. *If $D = \sum n_i E_i$ is an icoct, D' an effective divisor on X s.t. $D' \cdot E_i = 0$ for all i, then $D' = nD + D''$ where $n \geq 0, D''$ an effective divisor disjoint from D.*

Proof. Let n be the largest integer s.t. $D' - nD \geq 0$, and let $D'' = D' - nD, L = \mathcal{O}_D(D'')$. $\exists$ an exact sequence

$$0 \to \mathcal{O}_X(D'' - D) \to \mathcal{O}_X(D'') \to \mathcal{O}_D(D'') = L \to 0 \tag{3}$$

Let $s \in H^0(X, \mathcal{O}_X(D''))$ be s.t. $\operatorname{div}_X(s) = D''$. Since $D'' - D = D - (n+1)D$ is not effective, s doesn't come from $H^0(\mathcal{O}_X(D''-D))$, so its image in $H^0(\mathcal{O}_D(D''))$ is nonzero. But $\deg(L|_{E_i}) = D'' \cdot E_i = (D' - nD)\cdot E_i = 0 \implies L \cong \mathcal{O}_D \implies s(x) \neq 0 \,\forall\, x \in D$, so that the support of D'' must be disjoint from that of D. □

Theorem 1. *Let X be a minimal surface with $K^2 = 0$ and $K \cdot C \geq 0$ for all curves on X. If D is an icoct on X, $\exists$ an elliptic or quasielliptic fibration $f : X \to B$ on X obtained from the Stein factorization of $\phi_{|nD|} : X \to \mathbb{P}(H^0(\mathcal{O}_X(nD))^\vee)$ for some $n > 0$.*

Proof. Idea: use D and K to get an elliptic/quasielliptic fibration. Then show that the fiber must be a multiple of D.

Case 1: $p_g = 0$. or $n \geq 0$, we have the exact sequence

$$0 \to \mathcal{O}_X(nK + (n-1)D) \to \mathcal{O}_X(nK + nD) \to \mathcal{O}_D \to 0 \tag{4}$$

obtained by tensoring $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ by $n(K+D)$ and using $\mathcal{O}_X(nK+nD) \otimes \mathcal{O}_D \cong \omega_D^{\otimes n} \cong \mathcal{O}_D$ since D is an icoct. We claim that

$$H^2(\mathcal{O}_X(nK + (n-1)D)) = H^0(-(n-1)(K+D))0 \tag{5}$$

for $n \geq 2$. To see this, note that if $\Delta \in \left|\frac{m}{n}(K+D)\right|$ for $m > 0$, then either $\Delta = 0 \implies mK \sim -mD \implies K \cdot H = -D \cdot H < 0$ for an ample divisor H, giving a contradiction, or $\Delta > 0$ with a similar contradiction. Also, $H^2(\mathcal{O}_D) = 2$ since D has support of dimension 1, implying that $H^2(\mathcal{O}_X(nK+nD)) = 0$, and $H^1(\mathcal{O}_D) = H^0(\omega_D) = H^0(\mathcal{O}_D) \neq 0$ gives $H^1(\mathcal{O}_X(nK+nD)) \neq 0$. We know from Riemann-Roch that

$$\begin{aligned} \chi(\mathcal{O}_X(nK+nD)) &= \chi(\mathcal{O}_X) + \frac{1}{2}(nK+nD)(nK+nD-K) \\ &= \chi(\mathcal{O}_X) = 1 - q \end{aligned} \tag{6}$$

(since $p_g = 0$). Noether's formula states that

$$12 - 12q = 12 - 12q - 12p_g = K^2 + 2 - 2b_1 + b_2 \tag{7}$$

with $b_1 = 2q$ since the irregularity $\Delta = 0$ because $p_g = 0$. So

$$10 - 8q = b_2 \geq 1 \implies q \leq 1 \implies \chi(\mathcal{O}_X) = 0, 1 \tag{8}$$

and $\chi(\mathcal{O}_X(nK+nD)) = 0$ or 1 for $n \geq 2$. Since $H^1(\mathcal{O}_X(nK+nD)) \neq 0$ and $H^2(\mathcal{O}_X(nK+nD)) = 0$, we must have $H^0(\mathcal{O}_X(nK+nD)) \neq 0$ for $n \geq 2$. So $\exists\, D_n \in |nK+nD|$. As before, we see that $D_n \neq 0$.

We claim that D_n is of canonical type. Letting $D = \sum n_i E_i$, we find that

$$D_n \cdot E_i = n(K \cdot E_i) + n(D \cdot E_i) = 0 \tag{9}$$

This implies that $D_n = aD + \sum k_j F_j$ for some $a \geq 0, k_j > 0$ integers, F_j distinct irreducible curves that don't intersect D. Now $K \cdot F_j \geq 0$, and by our hypothesis $(\sum k_j F_j) \cdot K \geq 0$. But it equals $K \cdot nK + nD - D = 0$, so $K \cdot F_j = 0$ for all j. Finally,

$$D_n \cdot F_j = n(K \cdot F_j) + n(D \cdot F_j) = 0 \tag{10}$$

so D_n is of canonical type.

Now, D_n can't be a multiple of D for all n, For then $D_n = mD \implies nK \sim \lambda_n D$ for some integer λ_n for each $n \geq 2 \implies K = 3K \cdot 2K$ is a multiple of D, say $\lambda \cdot D = K$. If $\lambda < 0$, this contradicts $K \cdot H \geq 0$. If $\lambda \geq 0$, then $|K| = |\lambda D| = \varnothing$ which contradicts $p_g = 0$. So $\exists$ a curve of canonical type D' on X s.t. removing the multiple of D and decomposing to get an icoct, we get something disjoint from D. So let D' be an icoct, disjoint from D. Then

$$0 \to \mathcal{O}_X(2K + D + D') \to \mathcal{O}_X(2K + 2D + 2D') \to \mathcal{O}_D \oplus \mathcal{O}_{D'} \to 0 \tag{11}$$

(using $\omega_D \cong \mathcal{O}_D, \omega_{D'} \cong \mathcal{O}_{D'}$). As before, we can show that $H^2(\mathcal{O}_X(2K+D+D')) = 0$, and therefore $H^2(\mathcal{O}_X(2K+2D+2D')) = 0$. So $\chi(\mathcal{O}_X(2K+2D+2D')) = \chi(\mathcal{O}_X) = 0$ or 1, while $h^1(\mathcal{O}_X(2K+2D+_2D')) \geq 2$ (because $h^1(\mathcal{O}_D), h^1(\mathcal{O}_{D'}) \geq 1$) implies that $h^0(\mathcal{O}_X(2K + 2D + 2D')) \geq 0$. Now, take

$$\Delta \in |2K + 2D + 2D'|, \Delta > 0, \Delta^2 = 0, \dim |\Delta| \geq 1 \tag{12}$$

Since D, D' are of canonical type, so is Δ (easy exercise).

We now claim that $|\Delta|$ is composed with a pencil (i.e. it gives a map to a curve). To see this, let C be the fixed part of $|\Delta|$, then since Δ is of canonical type, we get $(\Delta - C)^2 \leq 0$ (the self-intersection of a divisor supported on a curve of canonical type is ≤ 0). So the rational map

$$\phi_{|\Delta|} : X \to \phi_{|\Delta|}(X) = B \subset |\Delta| \tag{13}$$

is defined everywhere (else would have $(\Delta - C)^2 > 0$. Use $C_1 \cdot C_2 = \tilde{C}_1 \cdot \tilde{C}_2 + m_1 m_2$ for a single blowup at p if C_1, C_2 pass through p with multiplicity m_1, m_2 and apply to two elements of $\widetilde{\Delta - C}$ with zero intersection after the blowup). Since dim $|\Delta| \geq 1, B$ can't be a point. And it can't be a surface, else we would have $\Delta \setminus C = \phi^*(H) \implies ((\Delta - C)^2) > 0$. So Δ is composed with a pencil and $\phi_{|\Delta|}$ is a morphism. Now $\Delta \cdot D = D \cdot (2K + 2D + 2D') = 0$ and $D \cdot (\Delta - C) \geq 0$ and $D \cdot C \geq 0$ (write C as $\sum k_i E_i + F_i$, where F doesn't have any of the E_i as components). This forces $D \cdot (\Delta - C) = 0$. Since D is connected, it is contained in one of the fibers and $D^2 = 0$. We see that D is a rational multiple of one of the fibers of the Stein factorization $f : X \to B' \to B$. Since the gcd of the coefficients of D is 1, the fiber must be a positive integral multiple of D. It is easy to see that the genus of the fiber is 1, implying that it is an elliptic/quasielliptic fibration.

Case 2: $p_g > 0$. As before, it is enough to show that $\dim H^0(\mathcal{O}_X(\Delta)) \geq 2$ for some divisor Δ of canonical type. We'll show that $\exists n > 0$. s.t. $\dim H^0(\mathcal{O}_X(nD)) \geq 2$. Let $\mathcal{F}_n = \mathcal{O}_X(nD)/\mathcal{O}_X$. So we have

(14)
$$0 \to \mathcal{O}_X \to \mathcal{O}_X(nD) \to \mathcal{F}_n \to 0 \implies H^0(\mathcal{O}_X(nD)) \to H^0(\mathcal{F}_n) \to H^1(\mathcal{O}_X)$$

It is enough to show that $H^0(\mathcal{F}_n) \to \infty$ as $n \to \infty$ since the dimension of $H^1(\mathcal{O}_X)$ is fixed. Let $\mathcal{L} = \mathcal{F}_1 = \mathcal{O}_X(D)/\mathcal{O}_X$ (note that $\mathcal{F}_0 = 0$). Then

$\mathcal{L}$ is an invertible sheaf on D, and

$$0 \to \mathcal{F}_{n-1} \to \mathcal{F}_n \to \mathcal{O}_X((n+1)D)/\mathcal{O}_X(nD) \cong \mathcal{L}^n \to 0 \tag{15}$$

implies that $n \mapsto h^0(\mathcal{F}_n)$ is nondecreasing. By Riemann-Roch,

$$\chi(\mathcal{O}_X(nD)) = \chi(\mathcal{O}_X) \implies \chi(\mathcal{F}_n) = 0 \tag{16}$$

for all n. One finds that $H^2(\mathcal{O}_X(nD)) = 0$ for $n >> 0$ since $K - nD$ has $h^0 = 0$ (D is effective). Thus, $H^1(\mathcal{F}_n) \neq 0$ for $n >> 0$ since $h^2(\mathcal{O}_X) = p_g > 0$ and we have the exact sequence

$$H^1(\mathcal{F}_n) \to H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_X(nD)) \tag{17}$$

This implies that $h^0(\mathcal{F}_n) = h^1(\mathcal{F}_n) > 0$ for $n >> 0$. If the sequence of integers $\{h^0(\mathcal{F}_n)\}$ is bounded above, let n be the largest s.t. $h^0(\mathcal{F}_{n-1}) < h^0(\mathcal{F}_n)$. (There exists such an n because $h^0(\mathcal{F}_0) = 0, h^0(\mathcal{F}_n) > 0$ for $n >> 0$.) We must have $h^0(\mathcal{F}_n) = h^0(\mathcal{F}_{n+1}) = \cdots$, and we obtain a nonzero global section of $\mathcal{L}^n$ coming from $s \in H^0(\mathcal{F}_n)$ not in the image of $H^0(\mathcal{F}_{n-1})$. D is an icoct and $\mathcal{L}^n$ has degree 0 on every component of D, so $s|_D$ does not vanish anywhere on D. Supp$\,(\mathcal{F}_n) = D \implies s$ generates $\mathcal{F}_n$ as an $\mathcal{O}_X$-module at all points of X, and thus defines a surjection $\mathcal{O}_X \to \mathcal{F}_n = \mathcal{O}_X(nD)/\mathcal{O}_X$ with kernel $\mathcal{O}_X(-nD)$ and an isomorphism $\mathcal{O}_X/\mathcal{O}_X(-nD) \cong \mathcal{O}_X(nD)/\mathcal{O}_X$. The tensor power gives an isomorphism $\mathcal{O}_X/\mathcal{O}_X(-nD) \xrightarrow{\sim} \mathcal{O}_X(mnD)/\mathcal{O}_X((m-1)nD) = \mathcal{F}_{mn}/\mathcal{F}_{(m-1)n}$ for all $m > 1$. Now, we have

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X((m-1)nD) & \longrightarrow & \mathcal{F}_{(m-1)n} & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(mnD) & \longrightarrow & \mathcal{F}_{mn} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{F}_n & & \mathcal{F}_n & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{18}$$

implying

$$\begin{array}{ccccccc} H^1(\mathcal{F}_{(m-1)n}) & \xrightarrow{\alpha} & H^1(\mathcal{F}_{mn}) & \longrightarrow & H^1(\mathcal{F}_n) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ H^2(\mathcal{O}_X) & \xrightarrow{=} & H^2(\mathcal{O}_X) & & & & \\ \downarrow & & \downarrow & & & & \\ 0 = H^2(\mathcal{O}_X((m-1)nD) & & H^2(\mathcal{O}_X(mnD)) = 0 & & & & \end{array} \tag{19}$$

for $m >> 0$. So α is nonzero (because $H^2(\mathcal{O}_X)$ is nonzero), $h^1(\mathcal{F}_{mn}) > h^1(\mathcal{F}_n)$ for $m >> 0$, implying that $h^0(\mathcal{F}_{mn}) > h^0(\mathcal{F}_n)$, a contradiction. □

Theorem 2. *Let X be a minimal surface with $K^2 = 0, K \cdot C \geq 0 \,\forall$ curves C on X. Then either $2K \sim 0$ or X has an icoct.*

Proof. Next time. □