18.727 Topics in Algebraic Geometry: Algebraic Surfaces
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ALGEBRAIC SURFACES, LECTURE 18

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Let X be as from last time, i.e. equipped with maps $f : X \to B, g : X \to \mathbb{P}^1$. Assume $\mathrm{char}(k) \neq 2, 3$ and let $S = \{c \in \mathbb{P}^1 | F_c \text{ is multiple}\}$. If $c \in \mathbb{P}^1 \smallsetminus S, f_c : F'_c \to B$ is an étale morphism. Then we have the map $f_c^* : \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(F'_c)$, and $\mathrm{Pic}^0(F'_c)$ acts canonically on F'_c. Thus, we get an action $B \times F'_c \to F'_c$ for each $c \in \mathbb{P}^1 \smallsetminus S$, and thus actions

$$\sigma_0 : B \times g^{-1}(\mathbb{P}^1 \times S) \to g^{-1}(\mathbb{P}^1 \smallsetminus S), \sigma : B \times X \to X \tag{1}$$

Explicitly, if $b \in B, x \in F'_c \subset X$ with $c \in \mathbb{P}^1 \smallsetminus S$, then $b \cdot X = y$, where $f^*\mathcal{O}_B(b - b_0) \otimes \mathcal{O}_{F'_c}(s) = \mathcal{O}_{F'_c}(y)$. Here b_0 is a fixed base point on B, which acts as the zero element of the elliptic curve B. Apply the norm $N_{F'_c|B}$ to get $\mathcal{O}_B(nb - nb_0 + f(x)) - \mathcal{O}_B(f(y))$ where $n = \deg f_c = F_b \cdot F'_c$. We thus obtain commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{b} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[t_{nb}]{} & B \end{array} \tag{2}$$

(where t_{nb} is translation by nb) and

$$\begin{array}{ccc} B \times X & \xrightarrow{\sigma} & X \\ {\scriptstyle \mathrm{id}_B \times f}\downarrow & & \downarrow{\scriptstyle f} \\ B \times B & \xrightarrow[(b,b') \mapsto nb+b']{} & B \end{array} \tag{3}$$

Let $B_0 = F_{b_0}$ and $A_n = \mathrm{Ker}\, n_B : B \to B$ a group subscheme of B. We see that the fibers of f are invariant under the action of A_n on X. In particular, A_n acts on B_0. Denote this by $\alpha : A_n \to \mathrm{Aut}\,(B_0)$, where $\mathrm{Aut}\,(B_0)$ is the group scheme of automorphisms of B_0. The action of B on X gives $\tau : B \times B_0 \to X$, which completes the diagram

$$\begin{array}{ccc} B \times B_0 & \xrightarrow{\tau} & X \\ & \searrow^{n_B \circ \mathrm{pr}_1} & \downarrow f \\ & & B \end{array} \tag{4}$$

Note that we can't use b_0 for an arbitrary element of B_0, since we already used it for a base point of B_0. So replace it by $b \in B$ and $b' \in B_0$. On can check that $\tau(b, x) = \tau(b', x') \Leftrightarrow \sigma(b - b', x) = x'$. Thus, X is isomorphic to the quotient of $B \times B_0$ by the action of A_n given by $a \cdot (b, b') = (b + a, \alpha(a)(b'))$ for $a \in A_n, b \in B, b' \in B_0$. We can substitute the curve $B/\mathrm{Ker}\,(\alpha)$ for B to get the following theorem:

Theorem 1. *Every hyperelliptic surface X has the form $X = B_1 \times B_0/A$, where B_0, B_1 are elliptic curves, A is a finite group subscheme of B_1, and A acts on the product $B_1 \times B_0$ by $a(b, b') = (b + a, \alpha(a)(b'))$ for $a \in A, b \in B_1, b' \in B_0$, and $\alpha : A \to \mathrm{Aut}\,(B_0)$ an injective homomorphism. The two elliptic fibrations of X are given by*

$$f : B_1 \times B_0/A \to B_1/A = B, g : B_1 \times B_0/A \to B_0/\alpha(A) \cong \mathbb{P}^1 \tag{5}$$

We can classify these, using the structure of a group of automorphisms of an elliptic curve $\mathrm{Aut}\,(B_0) = B_0 \rtimes \mathrm{Aut}\,(B_0, 0)$ (the group of translations and the group of automorphisms fixing 0 respectively). Explicitly, we have that

$$\mathrm{Aut}\,(B_0, 0) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & j(B_0) \neq 0, 1728 \\ \mathbb{Z}/4\mathbb{Z} & j(B_0) = 1728, \quad i.e.\ B_0 \cong \{y^2 = x^3 - x\} \\ \mathbb{Z}/6\mathbb{Z} & j(B_0) = 0, \quad i.e.\ B_0 \cong \{y^2 = x^3 - 1\} \end{cases} \tag{6}$$

Now $\alpha(A)$ can't be a subgroup of translations, else $B_0/\alpha(A)$ would be an elliptic curve, not $\mathbb{P}^1$. Let $\alpha \in A$ be s.t. $\alpha(a)$ generates the cyclic group $\overline{\alpha(A)}$ in $\mathrm{Aut}\,(B_0)/B_0 \cong \mathrm{Aut}\,(B_0, 0)$. It is easy to see that $\alpha(a)$ must have a fixed point. Choose that point to be the zero point of B_0. Now $\alpha(A)$ is abelian, so is a direct product $A_0 \times \mathbb{Z}/n\mathbb{Z}$. A_0 is a subgroup of translations of B_0 and thus a finite subgroup scheme of B_0. Since A_0 and $\alpha(A)$ commute, we must have $A_0 \subset \{b' \in B_0 | \alpha(a)(b') = b'\}$. We thus have the following possibilities:

(a) $n = 2 \implies$ the fixed points are $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
(b) $n = 3 \implies$ the fixed points are $\mathbb{Z}/3\mathbb{Z}$
(c) $n = 4 \implies$ the fixed points are $\mathbb{Z}/2\mathbb{Z}$
(d) $n = 6 \implies$ the fixed points are $\{0\}$

We thus obtain the following classification (Bagnera-de Franchis):

(a1) $(B_1 \times B_0)/(\mathbb{Z}/2\mathbb{Z})$, with the generator a of $\mathbb{Z}/2\mathbb{Z} \subset B_1[2]$ acting on $B_1 \times B_0$ by $a(b_1, b_0) = (b_1 + a, -b_0)$.

(a2) $(B_1 \times B_0)/(\mathbb{Z}/2\mathbb{Z})^2$, with the generators a and g of $(\mathbb{Z}/2\mathbb{Z})^2 \subset (B_1[2])^2$ acting by $a(b_1, b_0) = (b_1 + a, -b_0)$, $g(b_1, b_0) = (b_1 + g, b_0 + c)$ for $c \in B_0[2]$.

(b1) $(B_1 \times B_0)/(\mathbb{Z}/3\mathbb{Z})$, with the generator a of $\mathbb{Z}/3\mathbb{Z} = B_1[3]$ (s.t. $\alpha(a) = \omega \in \mathrm{Aut}\,(B_0, 0)$ an automorphism of order 3 [only when $j(B_0) = 0$]) acting on $B_1 \times B_0$ by $a(b_1, b_0) = (b_1 + a, \omega(b_0))$.

(b2) $(B_1 \times B_0)/(\mathbb{Z}/3\mathbb{Z})^2$, with the generators a and g of $(\mathbb{Z}/3\mathbb{Z})^2 = (B_1[3])^2$ acting by $a(b_1, b_0) = (b_1 + a, \omega(b_0))$, $g(b_1, b_0) = (b_1 + g, b_0 + c)$ for $c \in B_0[3]$, is fixed by ω, i.e. $\omega(c) = c$.

(c1) $(B_1 \times B_0)/(\mathbb{Z}/4\mathbb{Z})$, with the generator a of $\mathbb{Z}/4\mathbb{Z} \subset B_1[4]$ (s.t. $\alpha(a) = i \in \mathrm{Aut}\,(B_0, 0)$ an automorphism of order 4 [only when $j(B_0) = 1728$]) acting on $B_1 \times B_0$ by $a(b_1, b_0) = (b_1 + a, i(b_0))$.

(c2) $(B_1 \times B_0)/(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, with the generators a and g of $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = B_1[4] \times B_1[2]$ acting by $a(b_1, b_0) = (b_1 + a, i(b_0))$, $g(b_1, b_0) = (b_1 + g, b_0 + c)$ for $c \in B_0[2]$.

(d) $(B_1 \times B_0)/(\mathbb{Z}/6\mathbb{Z})$, with the generator a of $\mathbb{Z}/6\mathbb{Z} = B_1[6]$ acting on $B_1 \times B_0$ by $a(b_1, b_0) = (b_1 + a, -\omega(b_0))$.

1. Classification (contd.)

Our first goal is to prove the following theorem:

Theorem 2. *Let X be a minimal surface. Then*

(a) $\exists$ *an integral curve C on X s.t.* $K \cdot C < 0 \Leftrightarrow \kappa(X) = -\infty \Leftrightarrow p_g = p_0 = 0 \Leftrightarrow p_{12} = 0$.

(b) $K \cdot C = 0$ *for all integral curves C on X (i.e. $K \equiv 0$)* $\Leftrightarrow \kappa(X) = 0 \Leftrightarrow 4K \sim 0$ *or* $6K \sim 0 \Leftrightarrow 12K \sim 0$.

(c) $K^2 = 0, K \cdot C \geq 0$ *for all integral curves C on X, and $\exists$ an integral curve C' with* $K \cdot C' > 0 \Leftrightarrow \kappa(X) = 1 \Leftrightarrow K^2 = 0, |4K|$ *or* $|6K|$ *contains a strictly positive divisor* $\Leftrightarrow K^2 = 0, |12K|$ *has a strictly positive divisor.*

(d) $K^2 > 0, K \cdot C \geq 0$ *for all integral curves C on X* $\Leftrightarrow \kappa(X) = 2$*, in which case* $|2K| = \varnothing$.

We already showed that the 4 classes (given by the first clause) are exhaustive and mutually exclusive. We also proved the equivalences in (a). As a preliminary, we need some results on elliptic and quasielliptic fibrations. Recall that an effective divisor $D = \sum_{i=1}^{r} n_i E_i > 0$ is said to be of canonical type if $K_i \cdot E_i = D \cdot E_i = 0 \,\forall\, i$ (if $X \to B$ is an elliptic/quasielliptic fibration, then every fiber has this property). If D is also connected and $gcd(n_1, \ldots, n_r) = 1$, then we say that D is an indecomposable curve or a divisor of canonical type.

Proposition 1. *Let $D = \sum n_i E_i > 0$ be an indecomposable curve of canonical type on a minimal surface X, and let L be an invertible $\mathcal{O}_D$ module. If* $\deg(L \otimes \mathcal{O}_{E_i}) = 0$ *for all i, then* $H^0(D, L) \neq 0$ *iff* $L \cong \mathcal{O}_D$*. Also,* $H^0(D, \mathcal{O}_D) \cong k$.

Proof. It is enough to show that every nonzero section s of $H^0(D, L)$ generates L, i.e. gives an isomorphism $\mathcal{O}_D \cong L$. Then $H^0(D, \mathcal{O}_D)$ is a field containing k and is finite dimensional over k. Since k is algebraically closed by assumption, we have the proposition. So let $s \in H^0(D, L)$ be nonzero, and let $s_i = s|_{E_i} \in H^0(E_i, L \otimes \mathcal{O}_{E_i})$. The fact that $\deg(L \otimes \mathcal{O}_{E_i}) = 0$ implies that either s_i is identically 0 on E_i or s_i doesn't vanish anywhere on E_i (i.e. it generates $L \otimes \mathcal{O}_{E_i}$). If s_i is identically 0 on E_i, then s_j must be 0 on E_i for every E_j that intersects E_i. This implies that s_j vanishes at a point of E_j and thus on all of E_j for all j by the connectedness of D. So if s doesn't vanish identically on E_i for all i, then s doesn't vanish anywhere on D, and we again have the desired isomorphism.

So suppose that s_i is identically 0 on E_i for every i. We'll show that $s \neq 0$ gives a contradiction. Let k_i be the order of vanishing of s_i along $E_i, 1 \leq k \leq n_i$. Whenever $k_i < n_i$, s defines a nonzero section of $L \otimes \mathcal{O}_X(-k_iE_i)/\mathcal{O}_X((-k_i+1)E_i)$. We claim that this section vanishes at every point $p \in E_i$ to order at least the intersection multiplicity $(E_i, \sum_{j \neq i} k_jE_j; p)$. To see this, note that locally, if E_i only intersects one component E_j, $j \neq i$ at p, we can let $A = \mathcal{O}_{X,p}$ and $t_i = 0, t_j = 0$ cut out E_i and E_j respectively at p. We obtain an exact sequence

(7)
$$\begin{array}{ccccccc} & & & & H^0(L) & \searrow & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & H^0(E_i, L \otimes \mathcal{O}_X(-k_iE_i) \otimes \mathcal{O}_{E_i}) & \longrightarrow & H^0(L \otimes \mathcal{O}_{(k_i+1)E_i}) & \longrightarrow & H^0(L \otimes \mathcal{O}_{k_iE_i}) \end{array}$$

from the exact sequence

$$0 \to \mathcal{O}_X(-k_iE_i) \otimes \mathcal{O}_{E_i} \to \mathcal{O}_{(k_i+1)E_i} \to \mathcal{O}_{k_iE_i} \to 0 \tag{8}$$

after tensoring by L. The local version is

(9)
$$\begin{array}{ccccccc} & & & & s \in A/(t_i^{n_i}t_j^{n_j}) & \searrow & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & A/t_i & \longrightarrow & A/t_i^{k_i+1} & \longrightarrow & A/t_i^{k_i} \to 0 \end{array}$$

We can write $s = t_i^{k_i}\alpha_i = t_j^{k_j}\alpha_j, \alpha_i, \alpha_j \in A$ since the order of vanishing of s along t_i is k_i. Since t_i, t_j is an A-regular sequence, we get $\alpha_i = t_i^{k_j}\beta, \alpha_j = t_i^{k_i}\beta$, for some $\beta \in A$. The section s is represented by

$$t_i^{k_i}t_j^{k_j}\beta = t_j^{k_j}\beta \mod t_i \tag{10}$$

in A/t_i to the left of the diagram. Then

$$\operatorname{ord}_P(t_j^{k_j}\beta) = \dim(A/(t_i, t_j^{k_j}\beta) \geq \dim(A/(t_i, t_j^{k_j}) = \text{int.mult.}(E_i, k_jE_j; P) \tag{11}$$

In general, one can use the Chinese remainder theorem to get the inequality for many points P. So if $k_i < n_i$ then we have

$$
\begin{aligned}
(t_i, \sum_{j\neq i} k_j E_j) &\leq \deg E_i(L \otimes \mathcal{O}_X(-k_i E_i) \otimes \mathcal{O}_{E_i}) \\
&\leq \deg\left(\mathcal{O}_X(-E_i)/\mathcal{O}_X(-2E_i)\right)^{k_i} = -k_i E_i^2 \leq 0
\end{aligned} \tag{12}
$$

On the other hand, if $k_i = n_i$, then $E_i \cdot D = 0$ gives $E_i \cdot \sum k_j E_j = -(E_i, \sum(n_j - k_j)E_j) \leq 0$ since $k_j \leq n_j$ and $E_i \cdot E_j \geq 0$. So letting $D_1 = \sum k_j E_j$, we have $D_1 \cdot E_i \leq 0$ for all i. But

$$
\begin{aligned}
(D_1, D) = \sum k_i(E_i, D) = 0 \\
&\implies D_1 \cdot E_i = 0 \,\forall\, i \\
&\implies D_1^2 = 0 \\
&\implies D_1 \text{ is a rational multiple of } D \\
&\implies D_1 = D \\
&\implies k_i = n_i \,\forall\, i \,(\text{since } k_i \leq n_i \text{ and } \gcd(\{n_i\}) = 1) \\
&\implies s \equiv 0
\end{aligned} \tag{13}
$$

a contradiction. □