18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 16

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1. K3 surfaces contd.

Lemma 1. *Let $f : X' \to X$ be an étale map of degree n between surfaces X and X'. Then $\chi(\mathcal{O}_{X'}) = n\chi(\mathcal{O}_X)$.*

Proof. (In fact, this is true for general projective varieties, as a consequence of Grothendieck-Riemann-Roch.) Since f is étale, $f^*(\Omega^1_{X/k}) = \Omega^1_{X'/k}$ from

$$0 \to f^*(\Omega^1_{X/k}) \to \Omega^1_{X'/k} \to \Omega^1_{X'|X} = 0 \tag{1}$$

with the latter equality following from X' being étale over X. Thus, $f^*T_X = T_{X'}$ and $c_2(X') = c_2(T_{X'}) = c_2(f^*T_X) = f^*c_2(X)$ where c_2 is the class of T_X in the Chow ring of X. Taking degrees of these zero-cycles, we get $c_2(X') = (\deg f)c_2(X) = nc_2(X)$. We further have $\omega_{X'} = f^*\omega_X$, $(\omega_{X'} \cdot \omega_{X'}) = \deg f(\omega_X \cdot \omega_X) = n(\omega_X \cdot \omega_X)$. By Noether's formula, $\chi(\mathcal{O}_{X'} = \frac{1}{12}[(\omega_{X'} \cdot \omega_{X'}) + c_2(X')] = \frac{n}{12}[(\omega_X \cdot \omega_X) + c_2(X)] = n\chi(\mathcal{O}_X)$. □

1.1. Examples of $K3$ surfaces.

(1) A smooth quartic in $\mathbb{P}^3$: $\omega_X = \mathcal{O}_X(4-3-1) = \mathcal{O}_X$. Check that $H^1(X, \mathcal{O}_X) = 0$.

(2) Similarly, a smooth complete intersection of 3 quadrics in $\mathbb{P}^5$ or a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^4$ give $K3$ surfaces.

(3) A double sextic, i.e. a double cover of $\mathbb{P}^2$ branched over the zero locus of a smooth sextic polynomial (e.g. $z^2 = f(x, y)$ for f a polynomial of degree 6).

(4) For char$k \neq 2$, we get Kummer surfaces: starting with an abelian surface A over k, let $i : A \to A$ be the involution $x \mapsto -x$, and note that there are 16 fixed points, namely the points of $A[2](\overline{k})$. Blow these up to get $\pi : \tilde{A} \to A$. $\tilde{A}$ has 16 exceptional curves of the first kind, and i extends to give an involution $\tilde{i}$ of $\tilde{A}$. Then $\tilde{A}/\{1, \tilde{i}\}$ is a nonsingular surface and is $K3$, called the *Kummer surface* of A.

   To see this, we first show that $Y = \tilde{A}/\{1, \tilde{i}\}$ is smooth. Let $x_i, i = 1, \ldots, 16$ be the fixed points, $F_i = \pi^{-1}(x_i)$ the corresponding exceptional divisors. Now $\phi : \tilde{A} \to Y$ is étale away from $\bigcup F_i$. So we need to show that Y is smooth for a point in the image of $\bigcup F_i$. The translation τ_{x_i} induces an isomorphism $\tilde{A} \to \tilde{A}$ taking F_o to F_i. So it is enough to take a point on F_0, say x. Let u, v be regular local parameters at 0. We can take u, v s.t. $i^*u = -u, i^*v = -v$. Let $x \in F_1$, and assume that U and $t = \frac{v}{u}$ is a regular system of parameters at x. Then $i^*(u) = -u, i^*(\frac{v}{u}) = \frac{v}{u}$, implying that $u^2, \frac{v}{u} = t$ is a regular system of parameters at $\phi(x)$ and Y is smooth.

Now, let's compute the canonical bundle. We can trivialize ω_A so that locally at 0, it is $du \wedge dv$. At $x \in F, \omega = du \wedge dv = du \wedge d(tu) = udu \wedge dt = \frac{1}{2}d(u^2) \wedge dt$. So we see from the fact that u^2, t are regular local parameters at $\phi(x)$ that the divisor of ω_Y is zero $\implies \omega_Y \cong \mathcal{O}_Y$. Now the images of the 16 exceptional divisors are $E_i = \phi(F_i)$ and satisfy $E_i^2 = -2, E_i \cdot E_j = 0$ for $i \neq j$. So we see that the Picard number $\rho(X) \geq 17$ (including the ample class). By the Igusa-Severi inequality, $b_2 \geq \rho \geq 17$. By our classification, $b_2 = 22$ and Y is $K3$.

1.2. **Some general theory of K3 surfaces.** Over $\mathbb{C}$, the Hodge diamond $h^{p,q} = h^{qp} = H^p(X, \Omega^q)$ has the form

$$\begin{matrix} 1 & 0 & 1 \\ \\ 0 & 20 & 0 \\ \\ 1 & 0 & 1 \end{matrix} \tag{2}$$

Recall that $H^n(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$. $H^2(X, \mathbb{Z})$ is of fundamental importance. It is torsion free, and a lattice under the pairing given by the cup product. This pairing is symmetric, even, and unimodular. By the Hirzebruch index theorem, its signature is $(3, 19)$. By Noether's classification, it equals $U^3 \oplus E_8(-1)^2$, where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$E_8(-1) = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & 1 & \\ & & & & 1 & -2 & 0 & \\ & & & & 1 & 0 & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix} \tag{3}$$

Remark. Aside on Milnor's theorem: any even unimodular lattice of signature (m, n) for $m, n > 0$ is isomorphic to a sum of copies of U and E_8 or U and

$E_8(-1)$, with $E_8 \oplus E_8(-1) \cong U^8$. Any odd unimodular lattice of signature (m, n) is isomorphic to $1^m \oplus \langle -1 \rangle^n$.

Now, $H^2(X, \mathbb{Z}) \otimes \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ is the Hodge decomposition, and the image of $NS(X)$ lies in the $H^{1,1}$ subspace (in fact, is $H^{1,1} \cap H^2(X, \mathbb{Z})$ via the Lefschetz $(1,1)$-theorem). More generally, the Hodge conjecture states that, for a smooth variety $X/\mathbb{C}$ of dimension d, $H^{pp} \cap H^{2p}(X, \mathbb{Z})$ is generated by algebraic classes for all $p \leq d$. The regular 2-form lies in $H^{2,0} = H^0(X, \Omega^2)$. It pairs to 0 with all algebraic classes. The space $H^2(X, \mathbb{R}) \cap (H^{2,0} \cap H^{0,2})$ has signature $(0, 2)$. Thus, $H^{1,1}_R = H^{1,1} \cap H^2(X, \mathbb{R})$ has signature $(1, 19)$ i.e. it's a Lorentzian space. In it we can consider $\{x \in H^{1,1}_{\mathbb{R}} \mid x^2 > 0\}$, which contains 2 components, V^+, V^- where V^+ is the component containing the ample divisor. It is partitioned into chambers under the action of the Weyl group, which is generated by reflections in the hyperplanes orthogonal to the roots, $\Delta(X) = \{x \in H^{1,1}_{\mathbb{Z}} \mid x^2 = -2\}$. The fundamental chamber containing the Kähler form or ample divisor class is called the Kähler cone C^-_X. It is the set of elements in the positive cone that have positive intersection with any nonzero effective divisor class.

Next, note that any isomorphism $X \to X'$ of $K3$ surfaces determines an effective Hodge isometry $H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$, i.e. one which respects the Hodge decomposition, sends $V^+(X') \to V^+(X)$, and sends effective divisor classes to effective divisor classes (i.e. sends $C^+_{X'} \to C^+_X$).

Theorem 1 (Strong Torelli)**.** *An effective Hodge isometry $\phi : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ induces a unique isometry $f : X \to X'$ s.t. $\phi = f^*$.*

Period map: given X, we have $[\omega_X] \in \mathbb{P}(L_{K3} \otimes \mathbb{C}) \cong \mathbb{P}^{21}$ and $\omega_X^2 = 0$, $\omega_X \cdot \overline{\omega}_X > 0$ by Hodge theory. This gives a point in a complex open subset of a quadric in $\mathbb{P}^{21}$, which is some 20-dimensional domain Ω. By Todorov, the period map is surjective. By Siu, every $K3$ surface is Kähler. The moduli space of all $K3$ is 20 dimensional, while the algebraic $K3$ ($K3 + \mathcal{L}$ with $\mathcal{L}^2 = d$) have 19 moduli, and the moduli space is a countable union of 19-dimensional spaces. See Pitaetski-Shapiro and Shafarevich, *A Torelli Theorem for Algebraic K3 Surfaces* for details.

1.3. **Elliptic fibrations.** A $K3$ surface has an elliptic fibration iff $\exists$ a vector $v \in NS(X)$ with $v^2 = 0, v \neq 0$. Idea: let v correspond to a line bundle L. Now apply Riemann-Roch to get

$$h^0(L) - h^1(L) + h^2(L) = \chi(\mathcal{O}_X) + \frac{1}{2}L(L - K) = \chi(\mathcal{O}_X) = 2 \tag{4}$$

with

$$h^2(L) = h^0(K - L) = h^0(-L) = 0 \tag{5}$$

This implies that L or $-L$ is effective. Assume WLOG L is effective (otherwise, replace v by $-v$), represented by a divisor D. Then $h^0(L) \geq 2$. In fact, $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$ gives

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(D)) \to H^0(D, \mathcal{O}_D(D)) \to H^1(X, \mathcal{O}_X) = 0 \tag{6}$$

implying that $h^0(\mathcal{L}) = 2$. Thus, we get a map $X \to \mathbb{P}^1$, and the fiber has class v, $F^2 = 0$. Since $2g(F) - 2 = F(F + K) = 0, g(F) = 1$. By Bertini, the general fiber is irreducible and smooth. This gives us an elliptic fibration. If we want a section, look for a class of an effective divisor O s.t. $O \cdot F = 1$.