18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 14

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1. Elliptic and Quasi-elliptic Surfaces

1.1. Preliminary theorems.

Theorem 1. *Let $f : X \to Y$ be a dominant morphism from an irreducible, smooth algebraic variety X to an algebraic variety Y s.t. $f^* : k(Y) \hookrightarrow k(X)$ is separable and $k(Y)$ is algebraically closed in $k(X)$. Then $\exists$ a nonempty open subset $V \subset Y$ s.t. $\forall y \in V$, the fiber $f^{-1}(y)$ is geometrically integral.*

Sketch of proof. $k(Y) \hookrightarrow k(X)$ separable, $k(Y)$ algebraically closed in $k(X)$, implies that the generic fiber is geometrically integral over $k(Y)$ (see Milne's Algebraic Geometry for instance).Then some geometric reasoning shows that $\exists$ an open set for which this is true (i.e. the set for which $f^{-1}(y)$ is not geometrically integral is a constructible set). For full proof, see Badescu, Algebraic Surfaces, p. 87-90. $\square$

Let K be an algebraic function field in one variable over a perfect field k (i.e. K is a finitely generated extension of k of transcendence degree 1) and $L \supset K$ is an extension of K s.t. K is algebraically closed in L.

Theorem 2. *L/K is separable.*

Corollary 1. *Let $f : X \to Y$ be a dominant morphism from an irreducible smooth variety X of dimension ≥ 2 to an irreducible curve Y s.t. $k(Y)$ is algebraically closed in $k(X)$. Then the fiber $f^{-1}(y)$ is geometrically integral for all but finitely many closed points $y \in Y$.*

Theorem 3. *Let $f : X \to Y$ be a dominant morphism of smooth irreducible varieties over an algebraically closed field of characteristic 0. Then $\exists$ a nonempty open $V \subset Y$ s.t. $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is a smooth morphism, i.e. for every $y \in V$, the fiber $f^{-1}(y)$ is geometrically smooth of dimension $\dim X - \dim Y$.*

Definition 1. *A surjective morphism $f : X \to B$ from a surface X to a smooth projective curve B is called a fibration.*

Remark. f is necessarily flat (since it is surjective and B is a curve). Thus, the arithmetic genus of the fibers F_b for $b \in B$ is constant.

Definition 2. *f is called a (relatively) minimal fibration if $\forall b \in B, F_b$ does not contain any exceptional curves of the first kind.*

Let $f : X \to B$ be a fibration s.t. $f_*\mathcal{O}_X = \mathcal{O}_B$. By Zariski's main theorem, the fibers of f are connected. The condition is equivalent (by the second corollary) to $k(B)$ being algebraically closed in $k(X)$. So all but finitely many fibers are integral curves.

Definition 3. *A fibration f is elliptic if f is minimal, $f_*\mathcal{O}_X = \mathcal{O}_B$, and almost all the fibers of f are smooth curves of genus 1 (i.e. elliptic curves). If the fibers are singular integral curves of arithmetic genus 1, f is called quasi-elliptic.*

Note. Suppose $f : X \to B$ is quasi-elliptic. If any fiber of f is smooth, then the morphism f is generically smooth, i.e. f is elliptic. So for f to be quasi-elliptic, all the fibers must be singular. By the third theorem, quasi-elliptic fibrations cannot exist in characteristic 0.

Proposition 1. *Quasi-elliptic fibrations only exist in characteristics 2 and 3. The general fiber of f is a rational projective curve with one singular point, an ordinary cusp.*

Proof. Let $b' \in B$ be a closed point s.t. F_b is integral. Then $p_a(F_b) = 1$, F_b singular $\implies$ F_b is a rational curve with exactly one singular point which is a node or a cusp. Let's see that we can't have nodes. Let Σ be the set of points $x \in X$ where f is not smooth. Remove the (finite number of) points $b_1, \dots, b_n \in B$ s.t. F_{b_i} is not integral, and set $\Sigma_0 = \Sigma \cap f^{-1}(B \setminus \{b_1, \dots, b_n\})$.

Choose $x \in \Sigma_0$, and let $b = f(x)$. Let t be a regular local parameter at b and u, v regular local parameters at $x \in X$. Then f is given locally by $f(u, v) \in k[[u, v]]$, a formal power series corresponding to the completed local homomorphism $k[[t] = \widehat{\mathcal{O}_{B,b}} \to \widehat{\mathcal{O}_{X,x}} = k[[u, v]]$. Since F_b is integral, we can choose u, v s.t. $f(u, v)$ has the form

(1) $f(u, v) = G(u, v)(u^2 + v^3)$ (cusp)
(2) $f(u, v) = G(u, v)uv$ (node)

with $G(0, 0) \neq 0$. We have in case (2)

$$\frac{\partial f}{\partial u} = G(u, v)v + \frac{\partial G}{\partial u}(u, v)uv = v(\text{unit}) \tag{1}$$

(since $G(0, 0) \neq 0$) and similarly $\frac{\partial f}{\partial v} = u(\text{unit})$. So $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = 0$ cuts out the single point $0, 0$ and x is an isolated point of Σ_0. But then f is smooth away from x in a neighborhood of x, so by properness $\exists$ an open set $V \subset B \setminus \{b_i\}$ containing B s.t. the restriction of f to $f^{-1}(V)$ is smooth, implying that f is not quasi-elliptic. Thus, f must have a cusp everywhere away from F_{b_i}. Then Σ_0 is the locus of all such cuspidal points on $f^{-1}(B \setminus \{b_i\})$ and $f(u, v) = G(u, v)(u^2 + v^3), G(0, 0) = 0$.

If char$k \neq 2$, then

$$\frac{\partial f}{\partial u} = u\left[2G(u,v) + v^3\frac{\partial G}{\partial u}(u,v)\right] = 0 \tag{2}$$

defines, near x, a smooth curve that contains Σ_0. Thus, $\frac{\partial f}{\partial u}$ is a local equation of Σ_0 at x and Σ_0 is smooth at each of its points. The restriction of f to Σ_0 is a bijection of Σ_0 onto $B \smallsetminus \{b_i\}$. The intersection number $\Sigma_0 \cdot F_b, b \in B \smallsetminus \{b_i\}$ is

$$\begin{aligned} \Sigma_0 \smallsetminus F_b|_\lambda &= \dim\left(\mathcal{O}_{\Sigma_0,x}/\mathfrak{m}_{B,b}\mathcal{O}_{\Sigma_0,x}\right) = \dim k[[u,v]]/(f, \frac{\partial f}{\partial u}) \\ &= \dim k[[u,v]]/(u^2+v^3,u) = \dim k[v]/v^3 = 3 \end{aligned} \tag{3}$$

The field extension $k(B) \hookrightarrow k(\Sigma_0)$ is finite, purely inseparable (since Σ_0 is irreducible) and so it has degree p^m for some $m \geq 0$ ($p = $ char$k$). Thus, $p^n = \Sigma_0 \cdot F_b = 3$ for any $b \in B \smallsetminus \{b_i\}$, implying that char$k = 3$. □

2. Elliptic surfaces

Let $f : X \to B$ have generic fiber a smooth elliptic curve.

Theorem 4. *If f is smooth, $\exists$ an étale cover B' of B s.t. $f' : X' = X \times_B B' \to B$ is trivial (i.e. is a product $B' \times F$.*

Sketch of proof. We have a J-map $B \to \mathbb{A}^1$ given by $b \mapsto j(F_b)$. Since B is complete, any map to $\mathbb{A}^1$ is constant, so the j-invariant is constant on fibers. Now eliminate automorphisms by rigidifying (i.e. use full level N structure, e.g. with $N \geq 4$). □

Now, we'll consider the case where we have at least one singular fiber. Given $f : X \to B$, Tate's algorithm computes the singular fibers of the Néron model (minimal proper regular model). Recall the Weierstrass equation for an elliptic curve:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{4}$$

If we consider an elliptic surface, $a_i \in k(B)$, we can work locally at $b \in B$ and use Tate's algorithm. For $B = \mathbb{P}^1$, $a_i = a_i(t) \in k(t)$. We can clear denominators by multiplying x, y by λ^2, λ^3 to make $a_i \in k[t]$. We find that the singular fibers C_p fall into the following Kodaira-Néron classification:

Type I_1 C_p is a nodal rational curve.

Type I_n C_p consists of n smooth rational curves meeting with dual graph $\widetilde{A_n}$, i.e. a chain of c curves forming an n-gon.

Type I_n^* C_p consists of $n+5$ smooth rational curves meeting with dual graph $\widetilde{D_{n+4}}$, i.e. a chain of $n+1$ curves (multiplicity 2) with two additional curves (multiplicity 1) attached at either end.

Type II C_p is a cuspidal rational curve.

Type III C_p consists of two smooth rational curves meeting with dual graph $\widetilde{A_1}$, i.e. they meet at one point to order 2.

Type IV C_p consists of three smooth rational curves meeting with dual graph $\widetilde{A_2}$, i.e. they intersect at one point.

Type IV* C_p consists of seven smooth rational curves meeting with dual graph $\widetilde{E_6}$

Type III* C_p consists of eight smooth rational curves meeting with dual graph $\widetilde{E_7}$

Type II* C_p consists of nine smooth rational curves meeting with dual graph $\widetilde{E_8}$

Here, $\widetilde{A_n}$ etc. are the *extended Dynkin diagrams* corresponding to the simple groups A_n etc. We present a rough idea of how to classify these: let the singular fiber be $\sum n_i E_i$.

(1) $K \cdot E_i = 0$ for all i.
(2) If $r = 1, E_1^2 = 0, p_a(E_i) = 1$.
(3) if $r \geq 2$, then for each $1 \leq i \leq r, E_i^2 = -2, E_i \cong \mathbb{P}^1$, and $\sum_{j \neq i} n_j E_j E_i = 2n_i$.

Proof. By exercise, $E_i^2 \leq 0$. If $E_i^2 = 0$, then E_i is a multiple of the fiber and the fiber is irreducible. If $E_i^2 < 0$, then $K \cdot E_i < 0$ is not possible (the genus formula shows that E_i would have to be exceptional), so $K \cdot E_i \geq 0$. Then

(5)
$$0 = 2g(F) - 2 = F \cdot (F + K) = 0 + F \cdot K = \sum n_i (K \cdot E_i) \implies K \cdot E_i = 0 \forall i$$
$$E_i^2 < 0 \implies g(E_i) = 0, E_i^2 = -2, 0 = E_i \cdot F = \sum_{j \neq i} n_j (E_i E_j) - 2n_i$$

by the genus formula, giving the desired result. □

We use this information to bound the graphs that can arise and classify the singular fibers (see Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, Ch. IV).