18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 11

Recall from last time that we defined the group scheme $\underline{\mathrm{Pic}}_X$ over k as well as the group scheme $\underline{\mathrm{Pic}}^0_X$, which is the connected component of 0 (i.e. $\mathcal{O}_X$) in $\underline{\mathrm{Pic}}_X$ (and is a proper scheme over k). Now, let L be a line bundle in the class corresponding to the universal element. L is a line bundle on $X \times \mathrm{Pic}^0_X$. Choose a closed point x of X and let $M = L|_{\{x\}\times \underline{\mathrm{Pic}}^0_X}$. Then replace L by $L \otimes (p_2^* M)^{-1}$ so that we get $L|_{\{x\}\times \underline{\mathrm{Pic}}^0_X} \cong \mathcal{O}_{\underline{\mathrm{Pic}}^0_X}$ and, for every closed point $a \in \mathrm{Pic}^0_X$, the line bundle $L_a = L_{X\times\{a\}}$ is algebraically equivalent to 0. Such an L is called a Poincaré line bundle on $X \times \mathrm{Pic}^0_X$. Given a choice of basepoint a, it is unique up to isomorphism. Now, note further that the Zariski tangent space at 0 of Pic^0_X is canonically isomorphic to $H^1(X, \mathcal{O}_X)$ and $\underline{\mathrm{Pic}}^0_X$ is a commutative group scheme. If it is reduced, then it is an abelian variety. If $\mathrm{char}(k) = 0$, it is automatically reduced (by a theorem of Grothendieck-Cartier).

Theorem 1. *Let X be a surface, $q = h^1(X, \mathcal{O}_X)$ its irregularity, s the dimension of the Picard variety of X. Let b_1 be the first Betti number $= h^1_{ét}(X, \mathbb{Q}_\ell)$. Then $b_1 = 2s$ and $\Delta = 2q - b_1 = 2(q-s)$ lies between 0 and $2p_g$, while $\Delta = 0$ if* $\mathrm{char}(k) = 0$.

Proof. Note that, for ℓ relatively prime to $p = \mathrm{char}(k), \ell >> 0$

$$(1)\qquad \begin{aligned}(\mathbb{Z}/\ell\mathbb{Z})^{b_1} &= H^1_{ét}(X, \mathbb{Z}/\ell\mathbb{Z}) = \{a \in \mathrm{Pic}\, X | \ell \cdot a = 0\} \\ &= \{a \in \mathrm{Pic}^0 X | \ell \cdot a = 0\} = (\mathbb{Z}/\ell\mathbb{Z})^{2s}\end{aligned}$$

where the second equality follows from Kummer theory on $0 \to \mu_\ell \to \mathbb{F}_m \xrightarrow{\ell} \mathbb{F}_m \to 0$, the second from the fact that $\mathrm{Pic}/\mathrm{Pic}^0$ is finitely generated, so the torsion group is finite and ℓ can be chosen larger than the size of the torsion group, and the third because $\mathrm{Pic}^0(X)$ is the underlying abelian group of the Picard variety of X. The closed points of $\underline{\mathrm{Pic}}^0_X = (\underline{\mathrm{Pic}}^0_X)_{\mathrm{red}}$, so $b_1 = 2s$.

Now,

$$(2)\qquad \Delta = 2q - b_1 = 2(q-s) = 2\dim T_{\mathrm{Pic}^0_X, 0} - \dim T_{(\underline{\mathrm{Pic}}^0_X)_{\mathrm{red}},0} \geq 0$$

and

$$(3)\qquad q - s = \dim H^1(X, \mathcal{O}_X) - \dim (\cap_{i=1}^\infty \mathrm{Ker}\, \beta_i)$$

where the β_i are the *Bockstein homomorphisms* defined inductively by

$$(4)\qquad \beta_1 : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X), \beta_i : \mathrm{Ker}\, \beta_{i-1} \to \mathrm{coker}\, \beta_{i-1}$$

Thus, $q - s \le \dim\left(\cup_{i=1}^{\infty} \mathrm{Im}\,\beta_i\right) \le h^2(X, \mathcal{O}_X) = p_g$. In characteristic 0, proper group schemes of finite type are reduced, so $\underline{\mathrm{Pic}}_X^0$ is already an abelian variety. $\square$

0.1. **The Albanese Variety.** Let X be a smooth projective variety, $x_0 \in X$ a fixed closed point. A pair (A, α) consisting of an abelian variety and a morphism $\alpha : X \to A$ s.t. $\alpha_{x_0} = 0$ is called the *Albanese variety* of X. For every morphism $f : X \to B$ s.t. B is an abelian variety and $f(x_0) = 0$, $\exists$ a unique homomorphism of abelian varieties $g : A \to B$ s.t. the diagram below commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ {\scriptstyle \alpha}\big\downarrow & \nearrow_{g} & \\ A & & \end{array} \tag{5}$$

Note that a rigidity theorem for abelian varieties implies that any morphism (as varieties) $g' : A \to B$ is of the form $g'(a) = g(a) + b$ where $g : A \to B$ is a homomorphism of abelian varieties and $b = g'(0) \in B$. Thus, we can formulate the definition without the closed point x_0, where we say that there exists a unique homomorphism $g : A \to B$ s.t. $g \circ \alpha = f$. It is clearly unique if it exists.

For existence, let X be a smooth projective variety, and let $P(X)$ be the reduced Picard variety of X, and $P(X)^\vee$ its dual abelian variety. Then $\underline{\mathrm{Pic}}_{P(X)}^0 = P(X)^\vee$ (for an abelian variety, Pic^0 is automatically reduced). We have a universal Poincaré line bundle L on $X \times \underline{\mathrm{Pic}}_X^0$ and therefore on the reduced subscheme $X \times P(X)$. Let $\mu : X \times P(X) \to X \times P(X)$ be the switch $(y, x) \mapsto (x, y)$. $\mu^* L$ is a line bundle on $P(X) \times X$ and therefore comes from the Poincaré bundle on $P(X) \times P(X)^\vee$ by a map $X \to P(X)^\vee$ (by the universal property of $\mathrm{Pic}^0_{P(X)}$). One can check that this gives $P(X)^\vee$ as the Albanese variety of X using general nonsense, so $\mathrm{Alb}\,(X)$ exists and is unique up to unique isomorphism. Furthermore, it is dual to the Picard variety of X.

Note: If X is a smooth projective curve, then $\mathrm{Pic}^0(X)$ is reduced and carries a principal polarization, so $P(X)^\vee \cong P(X) \cong \mathrm{Pic}^0_X$ is the Jacobian of X. For a surface, we showed that the dimension of the Albanese variety is $\le q$, with equality holding $\Leftrightarrow \Delta = 0$ (e.g. if $\mathrm{char}(k) = 0$ or if $p_g = 0$).

If $k = \mathbb{C}$, there is an explicit way to see the Albanese variety. We have a map $i : H_1(X, \mathbb{Z}) \to H^0(X, \Omega_X^1)^*$ defined by $\langle i(\gamma), \omega \rangle = \int_\gamma \omega$. The image of i is a lattice in $H^0(X, \Omega_X^1)^*$, and the quotient is an abelian variety (a priori a complex torus, but a Riemann form exists). It is $\mathrm{Alb}\,(X)$, and is functorial in X, i.e.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle \alpha_X}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_Y} \\ \mathrm{Alb}\,(X) & \xrightarrow[\exists!]{} & \mathrm{Alb}\,(Y) \end{array} \tag{6}$$

It follows that the image of X in $\mathrm{Alb}\,(X)$ generates the abelian variety (else the subvariety that X generates inside $\mathrm{Alb}\,(X)$ would satisfy the universal property). In particular, if $\mathrm{Alb}\,(X) \neq 0$, $\alpha(X)$ is not a point, and if $X \to Y$ is a surjection, so is $\mathrm{Alb}\,(X) \to \mathrm{Alb}\,(Y)$. Over $\mathbb{C}$, our construction gives us an isomorphism $\alpha_* : H_1(X, \mathbb{Z}) \to H_1(\mathrm{Alb}\,(X), \mathbb{Z})$, so the inverse image under α of any étale covering of $\mathrm{Alb}\,(X)$ is connected. All Abelian coverings are obtained in this way.

For now, assume that $\mathrm{char}(k) = 0$.

Proposition 1. *Let X be a surface, $\alpha : X \to \mathrm{Alb}\,(X)$ the Albanese map. Suppose $\alpha(X)$ is a curve C. Then C is a smooth curve of genus q, and the fibers of α are connected.*

We first prove the following lemma:

Lemma 1. *Suppose α factors as $X \xrightarrow{f} T \xrightarrow{j} \mathrm{Alb}\,(X)$ with f surjective. Then $\tilde{j} : \mathrm{Alb}\,(T) \to \mathrm{Alb}\,(X)$ is an isomorphism.*

Proof. The functoriality of Alb gives a surjective morphism $\mathrm{Alb}\,(X) \to \mathrm{Alb}\,(T)$ (since $X \to T$ is surjective), along with a commutative diagram

(7)
$$\begin{array}{ccccc} X & \xrightarrow{f} & T & & \\ \downarrow{\scriptstyle \alpha_x} & & \downarrow{\scriptstyle \alpha_T} & \searrow & \\ \mathrm{Alb}\,(X) & \xrightarrow[f]{\cong} & \mathrm{Alb}\,(T) & \xrightarrow[\tilde{j}]{} & \mathrm{Alb}\,(X) \end{array}$$

$\tilde{j} \circ f$ is the identity by the universal property, so $\tilde{j}$ must be an isomorphism. ☐