10. The isomorphism theorems

We have already seen that given any group G and a normal subgroup H, there is a natural homomorphism $\phi\colon G \longrightarrow G/H$, whose kernel is H. In fact we will see that this map is not only natural, it is in some sense the only such map.

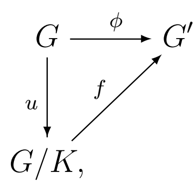
Theorem 10.1 (First Isomorphism Theorem)**.** *Let $\phi\colon G \longrightarrow G'$ be a homomorphism of groups. Suppose that ϕ is onto and let H be the kernel of ϕ.*

Then G' is isomorphic to G/H.

Proof. By the universal property of a quotient, there is a natural homorphism

$$f\colon G/H \longrightarrow G'.$$

As f makes the following diagram commute,



it follows that f is surjective. It remains to prove that f is injective. Suppose that x is in the kernel of f. Then x has the form gH and by definition of f, $f(x) = \phi(g)$. Thus g is in the kernel of ϕ and so $g \in H$. In this case $x = H$, the identity of G/H. So the kernel of f is trivial and f is injective. Hence f is an isomorphism. □

Definition 10.2. *Let G be a group and let H and K be two subgroups of G.*

$H \vee K$ denotes the subgroup generated by the union of H and K.

In general, it is hard to identify $H \vee K$ as a set. However,

Theorem 10.3 (Second Isomorphism Theorem)**.** *Let G be a group, let H be a subgroup and let N be a normal subgroup. Then*

$$H \vee N = HN = \{\, hn \mid h \in H, n \in N \,\}.$$

Furthermore $H \cap N$ is a normal subgroup of H and the two groups $H/H \cap N$ and HN/N are isomorphic.

Proof. The pairwise products of the elements of H and N are certainly elements of $H \vee N$. Thus the RHS of the equality above is a subset of the LHS. The RHS is clearly non-empty and so it suffices to prove that the RHS is closed under products and inverses.

Suppose that x and y are elements of the RHS. Then $x = h_1n_1$ and $y = h_2n_2$, where $h_i \in H$ and $n_i \in N$. Now $h_2^{-1}n_1h_2 = n_3 \in N$, as N is normal in G. So $n_1h_2 = h_2n_3$. In this case

$$\begin{aligned} xy &= (h_1n_1)(h_2n_2) \\ &= (h_1(n_1h_2)n_2 \\ &= (h_1(h_2n_3)n_2 \\ &= (h_1h_2)(n_3n_2), \end{aligned}$$

which shows that xy has the correct form. On the other hand, suppose $x = hn$. Then $hnh^{-1} = m \in N$ as N is normal and so $hn^{-1}h^{-1} = m^{-1}$. In this case

$$\begin{aligned} x^{-1} &= n^{-1}h^{-1} \\ &= hm^{-1}, \end{aligned}$$

so that x^{-1} is of the correct form.

Hence the first statement. Let $H \longrightarrow HN$ be the natural inclusion. As N is normal in G, it is certainly normal in HN, so that we may compose the inclusion with the natural homomorphism to get a homomorphism

$$H \longrightarrow HN/N.$$

This map sends h to hN.

Suppose that $x \in HN/N$. Then $x = hnN = hN$, where $h \in H$. Thus the homorphism above is clearly surjective. Suppose that $h \in H$ belongs to the kernel. Then $hN = N$ the identity coset, so that $h \in N$. Thus $h \in H \cap N$. The result then follows by the First Isomorphism Theorem applied to the map above. □

It is easy to prove the Third isomorphism Theorem from the First.

Theorem 10.4 (Third Isomorphism Theorem)**.** *Let $K \subset H$ be two normal subgroups of a group G.*

Then

$$G/H \simeq (G/K)/(H/K).$$

Proof. Consider the natural map $G \longrightarrow G/H$. The kernel, H, contains K. Thus, by the universal property of G/K, it follows that there is a homomorphism $G/K \longrightarrow G/H$.

This map is clearly surjective. In fact, it sends the left coset gK to the left coset gH. Now suppose that gK is in the kernel. Then the left coset gH is the identity coset, that is, $gH = H$, so that $g \in H$. Thus the kernel consists of those left cosets of the form gK, for $g \in H$, that is, H/K. The result now follows by the first Isomorphism Theorem. □

Recall that a subgroup is normal if it is invariant under conjugation. Now conjugation is just a special case of an automorphism of G.

Definition 10.5. *Let G be a group and let H be a subgroup. We say that H is a characteristic subgroup of G, if for every automorphism ϕ of G, $\phi(H) = H$.*

It turns out that most of the *general* normal subgroups that we have defined so far are all in fact characteristic subgroups.

Lemma 10.6. *Let G be a group and let $Z = Z(G)$ be the centre.*
Then Z is characteristically normal.

Proof. Let ϕ be an automorphism of G. We have to show $\phi(Z) = Z$.
Pick $z \in Z$. Then z commutes with every element of G. Pick an element x of G. As ϕ is a bijection, $x = \phi(y)$, for some $y \in G$. We have

$$\begin{aligned} x\phi(z) &= \phi(y)\phi(z) \\ &= \phi(yz) \\ &= \phi(zy) \\ &= \phi(z)\phi(y) \\ &= \phi(z)x. \end{aligned}$$

As x is arbitrary, it follows that $\phi(z)$ commutes with every element of G. But then $\phi(z) \in Z$. Thus $\phi(Z) \subset Z$. Applying the same result to the inverse ψ of ϕ we get $\phi^{-1}(Z) = \psi(Z) \subset Z$. But then $Z \subset \phi(Z)$, so that $Z = \phi(Z)$. □

Definition 10.7. *Let G be a group and let x and y be two elements of G. $x^{-1}y^{-1}xy$ is called the* ***commutator of** x **and** y*.
The **commutator subgroup** *of G is the group generated by all of the commutators.*

Lemma 10.8. *Let G be a group and let H be the commutator subgroup.*
Then H is characteristically normal in G and the quotient group G/H is abelian. Moreover this quotient is universal amongst all all abelian quotients in the following sense:
Suppose that $\phi \colon G \longrightarrow G'$ is any homomorphism of groups, where G' is abelian. Then there is a unique homomorphism $f \colon G/H \longrightarrow G'$ such that $f \circ u = \phi$.

Proof. Suppose that ϕ is an automorphism of G and let x and y be two elements of G. Then

$$\phi(x^{-1}y^{-1}xy) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y).$$

The last expression is clearly the commutator of $\phi(x)$ and $\phi(y)$. Thus $\phi(H)$ is generated by the commutators, and so $\phi(H) = H$. Thus H is characteristically normal in G.

Suppose that aH and bH are two left cosets. Then

$$\begin{aligned}(bH)(aH) &= baH \\ &= ba(a^{-1}b^{-1}ab)H \\ &= abH = (aH)(bH).\end{aligned}$$

Thus G/H is abelian.

Suppose that $\phi\colon G \longrightarrow G'$ is a homomorphism, and that G' is abelian. By the universal property of a quotient, it suffices to prove that the kernel of ϕ must contain H.

Since H is generated by the commutators, it suffices to prove that any commutator must lie in the kernel of ϕ. Suppose that x and y are in G. Then $\phi(x)\phi(y) = \phi(y)\phi(x)$. It follows that $\phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y)$ is the identity in G' so that $x^{-1}y^{-1}xy$ is sent to the identity, that is, the commutator of x and y lies in the kernel of ϕ. □

Definition-Lemma 10.9. *Let G and H be any two groups.*

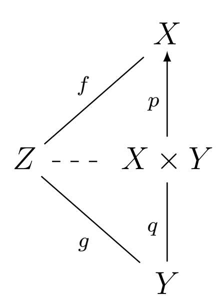
The product of G and H, denoted $G \times H$, is the group, whose elements are the ordinary elements of the cartesian product of G and H as sets, with multiplication defined as

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2).$$

Proof. We need to check that with this law of multiplication, $G \times H$ becomes a group. This is left as an exercise for the reader. □

Definition 10.10. *Let $\mathcal{C}$ be a category and let X and Y be two objects of $\mathcal{C}$. The categorical product of X and Y, denoted $X \times Y$, is an object together with two morphisms $p\colon X \times Y \longrightarrow X$ and $q\colon X \times Y \longrightarrow Y$ that are universal amongst all such morphisms, in the following sense.*

Suppose that there are morphisms $f\colon Z \longrightarrow X$ and $g\colon Z \longrightarrow Y$. Then there is a unique morphism $Z \longrightarrow X \times Y$ which makes the following diagram commute,



Note that, by the universal property of a categorical product, in any category, the product is unique, up to unique isomorphism. The proof proceeds exactly as in the proof of the uniqueness of a categorical quotient and is left as an exercise for the reader.

Lemma 10.11. *The product of groups is a categorical product.*

That is, given two groups G and H, the group $G \times H$ defined in (10.9) *satisfiees the universal property of* (10.10).

Proof. First of all note that the two ordinary projection maps $p\colon G \times H \longrightarrow G$ and $q\colon G \times H \longrightarrow H$ are both homomorphisms (easy exercise left for the reader).

Suppose that we are given a group K and two homomorphisms $f\colon K \longrightarrow G$ and $g\colon K \longrightarrow H$. We define a map $u\colon K \longrightarrow G \times H$ by sending k to $(f(k), g(k))$.

It is left as an exercise for the reader to prove that this map is a homomorphism and that it is the only such map, for which the diagram commutes. □

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