The Spectral Theorem for Hermitian Matrices

This is the proof that I messed up at the end of class on Nov 15.

For reference: A *Hermitian* means $A^* = A$. P *unitary* means $P^*P = I$.

Theorem. *Let A be a Hermitian matrix. There is a unitary matrix P such that $A' = P^*AP$ is a diagonal matrix.*

Some notation: We think of multiplication by the Hermitian matrix A as a linear operator on the standard Hermitian space $V = \mathbb{C}^n$, and we call that operator T. So A is the matrix of T with respect to the standard basis $\mathbf{E}$. The form on V is the standard Hermitian form $\langle v, w\rangle = v^*w$.

Let $\mathbf{B} = (v_1, ..., v_n)$ be a new basis, defined by $\mathbf{B} = \mathbf{E}P$, where P is unitary. The columns of P are the coordinate vectors of the vectors v_i. Since P is unitary, $P^* = P^{-1}$ and $P^*AP = P^{-1}AP$. So $A' = P^*AP$ is the matrix of the operator T with respect to $\mathbf{B}$. Therefore A and A' have the same eigenvalues.

We note that A' is Hermitian: $A'^* = (P^*AP)^* = P^*A^*P^{**} = P^*AP = A'$.

proof of the theorem. We choose an eigenvector v_1 of A and normalize its length to 1. Let W be the one-dimensional subspace $\text{Span}(v_1)$ of V. The matrix of the Hermitian form, restricted to W, is the 1×1 matrix whose unique entry is $\langle v_1, v_1\rangle = 1$. This matrix is invertible, so $V = W \oplus W^\perp$.

We choose an orthonormal basis $(v_2, ..., v_n)$ of $W^\perp$. Then $\mathbf{B} = (v_1, v_2, ..., v_n)$ will be an orthonormal basis of V. Since v_1 is an eigenvector, the matrix of T with respect to $\mathbf{B}$ will have the block form

$$A' = \begin{pmatrix} \lambda_1 & B \\ 0 & D \end{pmatrix},$$

where D is an $(n-1) \times (n-1)$ matrix, B and 0 are row and column vectors, respectively, and λ_1 is the eigenvalue of v_1. Since A' is Hermitian, $B = 0$ and D is Hermitian. It is the matrix that represents the operator T on $W^\perp$. By induction on dimension, we can choose the orthonormal basis $(v_2, ..., v_n)$ of $A^\perp$ so that D becomes diagonal. Then A' is also diagonal. □

Corollary. *The eigenvalues of a Hermitian matrix are real.*

proof. With notation as above, the eigenvalues of the matrix A are the same as those of A'. Since A' is Hermitian, its diagonal entries are real, and since A' is diagonal, its diagonal entries are the eigenvalues. □

Corollary. *The eigenvalues of a real symmetric matrix are real.*

proof. A real symmetric matrix is Hermitian. □

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